Gaussian binomial coefficients with negative arguments

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Abstract

Loeb showed that a natural extension of the usual binomial coefficient to negative (integer) entries continues to satisfy many of the fundamental properties. In particular, he gave a uniform binomial theorem as well as a combinatorial interpretation in terms of choosing subsets of sets with a negative number of elements. We show that all of this can be extended to the case of Gaussian binomial coefficients. Moreover, we demonstrate that several of the well-known arithmetic properties of binomial coefficients also hold in the case of negative entries. In particular, we show that Lucas' Theorem on binomial coefficients modulo p not only extends naturally to the case of negative entries, but even to the Gaussian case.

1 Introduction

Occasionally, the binomial coefficient $\binom{n}{k}$, with integer entries n and k, is considered to be zero when $k < 0$ (see Remark 1.9). However, as observed by Loeb [Loe92], there exists an alternative extension of the binomial coefficients to negative arguments, which is arguably more natural for many combinatorial or number theoretic applications. The q-binomial coefficients $\binom{n}{k}_q$ (often also referred to as Gaussian polynomials) are a polynomial generalization of the binomial coefficients that occur naturally in varied contexts, including combinatorics, number theory, representation theory and mathematical physics. For instance, if q is a prime power, then they count the number of k-dimensional subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$. We refer to the book [KC02] for a very nice introduction to the q-calculus. Yet, surprisingly, q-binomial coefficients with general integer entries have, to the best of our knowledge, not been introduced in the literature (Loeb [Loe92] does briefly discuss q-binomial coefficients but only in the case $k \geq 0$). The goal of this paper is to fill this gap, and to demonstrate that these generalized q-binomial coefficients are natural, by showing that they satisfy many of the fundamental combinatorial and arithmetic properties of the usual binomial coefficients. In particular, we extend Loeb's interesting combinatorial interpretation [Loe92] in terms of sets with negative numbers of elements. On the arithmetic side, we prove that Lucas' theorem can be uniformly generalized to both binomial coefficients and q-binomial coefficients with negative entries.

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In the context of q-series, it is common to introduce the q-binomial coefficient, for $n, k \geq 0$, as the quotient

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$ (1)

where $(a;q)_n$ denotes the q-Pochhammer symbol

$$(a;q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \quad n \geq 0.$$

In particular, $(a;q)_0 = 1$. It is not difficult to see that (1) reduces to the usual binomial coefficient in the limit $q \to 1$. In order to extend (1) to the case of negative integers n and k, we observe that the simple relation

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

can be used to extend the q-Pochhammer symbol to the case when $n < 0$. That is, if $n < 0$, it is common to define

$$(a;q)_n := \prod_{j=1}^{|n|} \frac{1}{1 - aq^{-j}}.$$

Note that $(q;q)_n = \infty$ whenever $n < 0$, so that (1) does not immediately extend to the case when n or k are negative. We therefore make the following definition, which clearly reduces to (1) when $n, k \geq 0$.

Definition 1.1. For all integers n and k,

$$\binom{n}{k}_q := \lim_{a \to q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}}.$$ (2)

Though not immediately obvious from (2) when n or k are negative, these generalized q-binomial coefficients are Laurent polynomials in q with integer coefficients. In particular, upon setting $q = 1$, we always obtain integers.

Example 1.2.

$$\binom{-3}{-5}_q = \lim_{a \to q} \frac{(a;q)_{-3}}{(a;q)_{-5}(a;q)_2} = \lim_{a \to q} \frac{\left(1 - \frac{a}{q^4}\right)\left(1 - \frac{a}{q^5}\right)}{(1-a)(1-aq)} = \frac{(1+q^2)(1+q+q^2)}{q^7}$$

In Section 2, we observe that, for integers n and k, the q-binomial coefficients are also characterized by the Pascal relation

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q,$$ (3)

provided that $(n, k) \neq (0, 0)$ (this exceptional case excludes itself naturally in the proof of Lemma 2.1), together with the initial conditions

$$\binom{n}{0}_q = \binom{n}{n}_q = 1.$$

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In the case $q = 1$, this extension of Pascal's rule to negative parameters was observed by Loeb [Loe92, Proposition 4.4].

Among the other basic properties of the generalized q-binomial coefficient are the following. All of these are well-known in the classical case $n, k \geq 0$. That they extend uniformly to all integers n and k (though, as illustrated by (3) and item (c), some care has to be applied when generalizing certain properties) serves as a first indication that the generalized q-binomial coefficients are natural objects. For (c), the sign function $\operatorname{sgn}(k)$ is defined to be 1 if $k \geq 0$, and -1 if $k < 0$.

Lemma 1.3. *For all integers n and k,*

(a) $\binom{n}{k}_q = q^{k(n-k)} \binom{n}{k}_{q^{-1}}$,

(b) $\binom{n}{k}_q = \binom{n}{n-k}_q$,

(c) $\binom{n}{k}_q = (-1)^k \operatorname{sgn}(k) q^{\frac{1}{2}k(2n-k+1)} \binom{k-n-1}{k}_q$,

(d) $\binom{n}{k}_q = \frac{1-q^n}{1-q^k} \binom{n-1}{k-1}_q$ *if $k \neq 0$.*

Properties (b) and (d) follow directly from the definition (2), while property (a) is readily deduced from (3) combined with (b). In the classical case $n, k \geq 0$, property (a) reflects the fact that the q-binomial coefficient is a self-reciprocal polynomial in q of degree $k(n-k)$. In contrast to that and as illustrated in Example 1.2, the q-binomial coefficients with negative entries are Laurent polynomials, whose degrees are recorded in Corollary 3.3.

The reflection rule (c) is the subject of Section 3 and is proved in Theorem 3.1. Rule (c) reduced to the case $q = 1$ is the main object in [Spr08], where Sprugnoli observed the necessity of including the sign function when extending the binomial coefficient to negative entries. Sprugnoli further realized that the basic symmetry (b) and the negation rule (c) act on binomial coefficients as a group of transformations isomorphic to the symmetric group on three letters. In Section 3, we observe that the same is true for q-binomial coefficients.

Note that property (d), when combined with (b), implies that, for $n \neq k$,

$$\binom{n}{k}_q = \frac{1-q^n}{1-q^{n-k}} \binom{n-1}{k}_q.$$

In particular, the q-binomial coefficient is a q-hypergeometric term.

Example 1.4. It follows from Lemma 1.3(c) that, for all integers k,

$$\binom{-1}{k}_q = (-1)^k \operatorname{sgn}(k) \frac{1}{q^{k(k+1)/2}}.$$

In Section 4, we review the remarkable and beautiful observation of Loeb [Loe92] that the combinatorial interpretation of binomial coefficients as counting subsets can be naturally extended to the case of negative entries. We then prove that this interpretation can be generalized to q-binomial coefficients. Theorem 4.5, our main result of that section, is a precise version of the following.

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Theorem 1.5. *For all integers n and k,*

$$\binom{n}{k}_q = \pm \sum_Y q^{\sigma(Y)-k(k-1)/2},$$

where the sum is over all k-element subsets Y of the n-element set X_n.

The notion of sets (and subsets) with a negative number of elements, as well as the definitions of σ and X_n, are deferred to Section 4. In the previously known classical case $n, k \geq 0$, the sign in that formula is positive, $X_n = \{0, 1, 2, \ldots, n-1\}$, and $\sigma(Y)$ is the sum of the elements of Y. As an application of Theorem 1.5, we demonstrate at the end of Section 4 how to deduce from it generalized versions of the Chu-Vandermonde identity as well as the (commutative) q-binomial theorem.

In Section 5, we discuss the binomial theorem, which interprets the binomial coefficients as coefficients in the expansion of $(x + y)^n$. Loeb showed that, by also considering expansions in inverse powers of x, one can extend this interpretation to the case of binomial coefficients with negative entries. Once more, we are able to show that the generalized q-binomial coefficients share this property in a uniform fashion.

Theorem 1.6. *Suppose that $yx = qxy$. Then, for all integers n, k,*

$$\binom{n}{k}_q = \{x^k y^{n-k}\}(x + y)^n.$$

Here, the operator $\{x^k y^{n-k}\}$, which is defined in Section 5, extracts the coefficient of $x^k y^{n-k}$ in the appropriate expansion of what follows.

A famous theorem of Lucas [Luc78] states that, if p is a prime, then

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1} \cdots \binom{n_d}{k_d} \pmod{p},$$

where n_i and k_i are the p-adic digits of the nonnegative integers n and k, respectively. In Section 6, we show that this congruence in fact holds for all integers n and k. In fact, in Section 7, we prove that generalized Lucas congruences uniformly hold for q-binomial coefficients.

Theorem 1.7. *Let $m \geq 2$ be an integer. Then, for all integers n and k,*

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'} \pmod{\Phi_m(q)},$$

where $n = n_0 + n'm$ and $k = k_0 + k'm$ with $n_0, k_0 \in \{0, 1, \ldots, m-1\}$.

Here, $\Phi_m(q)$ is mth cyclotomic polynomial. The classical special case $n, k \geq 0$ of this result has been obtained by Olive [Oli65] and Désarménien [Dés82].

We conclude this introduction with some comments on alternative approaches to and conventions for binomial coefficients with negative entries. In particular, we remark on the current state of computer algebra systems and how it could benefit from the generalized q-binomial coefficients introduced in this paper.

Remark 1.8. Using the gamma function, binomial coefficients can be introduced as

$$\binom{n}{k} := \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \tag{4}$$

for all complex n and k such that $n, k \notin \{-1, -2, \ldots\}$. This definition, however, does not immediately lend itself to the case of negative integers; though the structure of poles (and lack of zeroes) of the underlying gamma function is well understood, the binomial function (4) has a subtle structure when viewed as a function of two variables. For a study of this function, as well as a historical account on binomials, we refer to [Fow96]. A natural way to extend (4) to negative integers is to set

$$\binom{n}{k} := \lim_{\varepsilon \to 0} \frac{\Gamma(n+1+\varepsilon)}{\Gamma(k+1+\varepsilon)\Gamma(n-k+1+\varepsilon)}, \tag{5}$$

where n and k are now allowed to take any complex values. This is in fact the definition that Loeb [Loe92] and Sprugnoli [Spr08] adopt. (That the q-binomial coefficients we introduce in (2) reduce to (5) when $q = 1$ can be seen, for instance, from observing that the Pascal relation (3) reduces to the relation established by Loeb for (5).)

Remark 1.9. Other conventions for binomial coefficients with negative integer entries exist and have their merit. Most prominently, if, for instance, one insists that Pascal's relation (3) should hold for all integers n and k, then the resulting version of the binomial coefficients is zero when $k < 0$. On the other hand, as illustrated by the results in [Loe92] and this paper, it is reasonable and preferable for many purposes to extend the classical binomial coefficients (as well as its polynomial counterpart) to negative arguments as done here.

As an unfortunate consequence, both conventions are implemented in current computer algebra systems, which can be a source of confusion. For instance, SageMath currently (as of version 8.0) uses the convention that all binomial coefficients with $k < 0$ are evaluated as zero. On the other hand, recent versions of Mathematica (at least version 9 and higher) and Maple (at least version 18 and higher) evaluate binomial coefficients with negative entries in the way advertised in [Loe92] and here.

In version 7, Mathematica introduced the `QBinomial[n,k,q]` function; however, as of version 11, this function evaluates the q-binomial coefficient as zero whenever $k < 0$. Similarly, Maple implements these coefficients as `QBinomial(n,k,q)`, but, as of version 18, choosing $k < 0$ results in a division-by-zero error. We hope that this paper helps to adjust these inconsistencies with the classical case $q = 1$ by offering a natural extension of the q-binomial coefficient for negative entries.

2 Characterization via a q-Pascal relation

The generalization of the binomial coefficients to negative entries by Loeb satisfies Pascal's rule

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{6}$$

for all integers n and k that are not both zero [Loe92, Proposition 4.4]. In this brief section, we show that the q-binomial coefficients (with arbitrary integer entries), defined in (2), are also characterized by a q-analog of the Pascal rule. It is well-known that this is true for the familiar q-binomial coefficients when $n, k \geq 0$ (see, for instance, [KC02, Proposition 6.1]).

Lemma 2.1. *For integers n and k, the q-binomial coefficients are characterized by*

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q, \tag{7}$$

provided that $(n,k) \neq (0,0)$, together with the initial conditions

$$\binom{n}{0}_q = \binom{n}{n}_q = 1.$$

Observe that $\binom{0}{0}_q = 1$, while the corresponding right-hand side of (7) is $\binom{-1}{-1}_q + q^0 \binom{-1}{0}_q = 2 \neq 1$, illustrating the need to exclude the case $(n,k) = (0,0)$. It should also be noted that the initial conditions are natural but not minimal: the case $\binom{n}{0}_q$ with $n \leq -2$ is redundant (but consistent).

Proof. We note that the relation (7) and the initial conditions indeed suffice to deduce values for each q-binomial coefficient. It therefore only remains to show that (7) holds for the q-binomial coefficient as defined in (2). For the purpose of this proof, let us write

$$\binom{n}{k}_{a,q} := \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}},$$

and observe that, for all integers n and k,

$$\binom{n-1}{k}_{a,q} = \frac{1 - aq^{n-k-1}}{1 - aq^{n-1}} \binom{n}{k}_{a,q}$$

as well as

$$\binom{n-1}{k-1}_{a,q} = \frac{1 - aq^{k-1}}{1 - aq^{n-1}} \binom{n}{k}_{a,q}.$$

It then follows that

$$\binom{n}{k}_{a,q} = \binom{n-1}{k-1}_{a,q} + aq^{k-1} \frac{1 - q^{n-k}}{1 - aq^{n-k-1}} \binom{n-1}{k}_{a,q} \tag{8}$$

for all integers n and k. If $n \neq k$, then

$$\lim_{a \to q} \left[aq^{k-1} \frac{1 - q^{n-k}}{1 - aq^{n-k-1}} \right] = q^k,$$

so that (7) follows for these cases. On the other hand, if $n = k$, then $\binom{n-1}{k}_q = 0$, provided that $(n,k) \neq (0,0)$, so that (7) also holds in the remaining cases. $\square$

Remark 2.2. Applying Pascal's relation (7) to the right-hand side of Lemma 1.3(b), followed by applying the symmetry Lemma 1.3(b) to each q-binomial coefficient, we find that Pascal's relation (7) is equivalent to the alternative form

$$\binom{n}{k}_q = q^{n-k} \binom{n-1}{k-1}_q + \binom{n-1}{k}_q. \tag{9}$$

3   Reflection formulas

In [Spr08], Sprugnoli, likely unaware of the earlier work of Loeb [Loe92], introduces binomial coefficients with negative entries via the gamma function (see Remark 1.8). Sprugnoli then observes that the familiar negation rule

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$$

as stated, for instance, in [Knu97, Section 1.2.6], does not continue to hold when k is allowed to be negative. Instead, he shows that, for all integers n and k,

$$\binom{n}{k} = (-1)^k \operatorname{sgn}(k) \binom{k-n-1}{k}, \tag{10}$$

where $\operatorname{sgn}(k) = 1$ for $k \geq 0$ and $\operatorname{sgn}(k) = -1$ for $k < 0$. We generalize this result to the q-binomial coefficients. Observe that the result of Sprugnoli [Spr08] is immediately obtained as the special case $q = 1$.

Theorem 3.1. *For all integers n and k,*

$$\binom{n}{k}_q = (-1)^k \operatorname{sgn}(k) q^{\frac{1}{2}k(2n-k+1)} \binom{k-n-1}{k}_q. \tag{11}$$

Proof. Let us begin by observing that, for all integers n and k,

$$(a;q)_n (aq^n;q)_k = (a;q)_{n+k}. \tag{12}$$

Further, for all integers n,

$$(a;q)_n = (-a)^n q^{n(n-1)/2} (q^{-n+1}/a;q)_n. \tag{13}$$

Applying (12) and then (13), we find that

$$\frac{(a;q)_n}{(a;q)_{n-k}} = \frac{1}{(aq^n;q)_{-k}} = \frac{(-a)^k q^{\frac{1}{2}k(2n-k-1)}}{(q^{k-n+1}/a;q)_{-k}}.$$

By another application of (12),

$$\frac{1}{(q^{k-n+1}/a;q)_{-k}} = \frac{(1/a;q)_{k-n+1}}{(1/a;q)_{-n+1}} = \frac{(q^2/a;q)_{k-n-1}}{(q^2/a;q)_{-n-1}},$$

where, for the second equality, we used the basic relation $(a;q)_n = (1-a)(aq;q)_{n-1}$ twice for each Pochhammer symbol. Combined, we thus have

$$\frac{(a;q)_n}{(a;q)_{n-k}} = (-a)^k q^{\frac{1}{2}k(2n-k-1)} \frac{(q^2/a;q)_{k-n-1}}{(q^2/a;q)_{-n-1}}$$

for all integers n and k. Suppose we have already shown that, for any integer n,

$$\lim_{a \to q} \frac{(q^2/a;q)_n}{(a;q)_n} = \operatorname{sgn}(n). \tag{14}$$

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Then,

$$
\begin{aligned}
\binom{n}{k}_q &= \lim_{a \to q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}} \\
&= \lim_{a \to q} (-a)^k q^{\frac{1}{2}k(2n-k-1)} \frac{(q^2/a;q)_{k-n-1}}{(a;q)_k (q^2/a;q)_{-n-1}} \\
&= \operatorname{sgn}(k-n-1)\operatorname{sgn}(-n-1) \lim_{a \to q} (-a)^k q^{\frac{1}{2}k(2n-k-1)} \frac{(a;q)_{k-n-1}}{(a;q)_k (a;q)_{-n-1}} \\
&= (-1)^k \operatorname{sgn}(k) q^{\frac{1}{2}k(2n-k+1)} \binom{k-n-1}{k}_q .
\end{aligned}
$$

For the final equality, we used that

$$
\operatorname{sgn}(k-n-1)\operatorname{sgn}(-n-1) = \operatorname{sgn}(k)
$$

whenever the involved q-binomial coefficients are different from zero (for more details on this argument, see [Spr08, Theorem 2.2]).

It remains to show (14). The limit clearly is 1 if $n \geq 0$. On the other hand, if $n < 0$, then

$$
\begin{aligned}
\lim_{a \to q} \frac{(q^2/a;q)_n}{(a;q)_n} &= \lim_{a \to q} \frac{\left(1 - \frac{a}{q}\right)\left(1 - \frac{a}{q^2}\right) \cdots \left(1 - \frac{a}{q^n}\right)}{\left(1 - \frac{q}{a}\right)\left(1 - \frac{1}{a}\right) \cdots \left(1 - \frac{1}{aq^{n-2}}\right)} \\
&= \lim_{a \to q} \frac{\left(1 - \frac{a}{q}\right)}{\left(1 - \frac{q}{a}\right)} = -1,
\end{aligned}
$$

as claimed. $\qquad\square$

It was observed in [Spr08, Theorem 3.2] that the basic symmetry (Lemma 1.3(b)) and the negation rule (11) act on (formal) binomial coefficients as a group of transformations isomorphic to the symmetric group on three letters. The same is true for q-binomial coefficients. Since the argument is identical, we only record the resulting six forms for the q-binomial coefficients.

Corollary 3.2. *For all integers n and k,*

$$
\begin{aligned}
\binom{n}{k}_q &= \binom{n}{n-k}_q \\
&= (-1)^{n-k} \operatorname{sgn}(n-k) q^{\frac{1}{2}(n(n+1)-k(k+1))} \binom{-k-1}{n-k}_q \\
&= (-1)^{n-k} \operatorname{sgn}(n-k) q^{\frac{1}{2}(n(n+1)-k(k+1))} \binom{-k-1}{-n-1}_q \\
&= (-1)^k \operatorname{sgn}(k) q^{\frac{1}{2}k(2n-k+1)} \binom{k-n-1}{-n-1}_q \\
&= (-1)^k \operatorname{sgn}(k) q^{\frac{1}{2}k(2n-k+1)} \binom{k-n-1}{k}_q .
\end{aligned}
$$

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Proof. These equalities follow from alternately applying the basic symmetry from Lemma 1.3(b) and the negation rule (11). Moreover, for the fourth equality, we use that

$$- \operatorname{sgn}(n - k) \operatorname{sgn}(-n - 1) = \operatorname{sgn}(k)$$

whenever the involved q-binomial coefficients are different from zero (again, see [Spr08, Theorem 2.2] for more details on this argument). □

It follows directly from the definition (2) that the q-binomial coefficient $\binom{n}{k}_q$ is zero if $k > n \geq 0$ or if $n \geq 0 > k$. The third equality in Corollary 3.2 then makes it plainly visible that the q-binomial coefficient also vanishes if $0 > k > n$. Moreover, we can read off from Corollary 3.2 that the q-binomial coefficient is nonzero otherwise; that is, it is nonzero precisely in the three regions $0 \leq k \leq n$ (the classical case), $n < 0 \leq k$ and $k \leq n < 0$. More precisely, we have the following, of which the first statement is, of course, well-known (see, for instance, [KC02, Corollary 6.1]).

Corollary 3.3.

(a) If $0 \leq k \leq n$, then $\binom{n}{k}_q$ is a polynomial of degree $k(n - k)$.

(b) If $n < 0 \leq k$, then $\binom{n}{k}_q$ is $q^{\frac{1}{2}k(2n-k+1)}$ times a polynomial of degree $k(-n - 1)$.

(c) If $k \leq n < 0$, then $\binom{n}{k}_q$ is $q^{\frac{1}{2}(n(n+1)-k(k+1))}$ times a polynomial of degree $(-n - 1)(n - k)$.

In each case, the polynomials are self-reciprocal and have integer coefficients.

Observe that Corollary 3.2 together with the defining product (1), spelled out as

$$\binom{n}{k}_q = \frac{(1 - q^{k+1})(1 - q^{k+2}) \cdots (1 - q^n)}{(1 - q)(1 - q^2) \cdots (1 - q^{n-k})}$$

and valid when $0 \leq k \leq n$, provides explicit product formulas for all choices of n and k. Indeed, the three regions in which the binomial coefficients are nonzero are $0 \leq k \leq n$, $n < 0 \leq k$ and $k \leq n < 0$, and these three are permuted by the transformations in Corollary 3.2.

4 Combinatorial interpretation

For integers $n, k \geq 0$, the binomial coefficient $\binom{n}{k}$ counts the number of k-element subsets of a set with n elements. It is a remarkable and beautiful observation of Loeb [Loe92] that this interpretation (up to an overall sign) can be extended to all integers n and k by a natural notion of sets with a negative number of elements. In this section, after briefly reviewing Loeb's result, we generalize this combinatorial interpretation to the case of q-binomial coefficients.

Let U be a collection of elements (the "universe"). A set X with elements from U can be thought of as a map $M_X : U \to \{0, 1\}$ with the understanding that $u \in X$ if and only if $M_X(u) = 1$. Similarly, a multiset X can be thought of as a map $M_X : U \to \{0, 1, 2, \ldots\}$, in which case $M_X(u)$ is the multiplicity of an element u. In this spirit, Loeb introduces a *hybrid set* X as a map $M_X : U \to \mathbb{Z}$. We will denote hybrid sets in the form $\{\ldots | \ldots\}$, where elements with a positive multiplicity are listed before the bar, and elements with a negative multiplicity after the bar.

Example 4.1. The hybrid set $\{1, 1, 4|2, 3, 3\}$ contains the elements $1, 2, 3, 4$ with multiplicities $2, -1, -2, 1$, respectively.

A hybrid set Y is a subset of a hybrid set X, if one can repeatedly remove elements from X (here, removing means decreasing by one the multiplicity of an element with nonzero multiplicity) and thus obtain Y or have removed Y. We refer to [Loe92] for a more formal definition and further discussion, including a proof that this notion of being a subset is a well-defined partial order (but not a lattice).

Example 4.2. From the hybrid set $\{1, 1, 4|2, 3, 3\}$ we can remove the element 4 to obtain $\{1, 1|2, 3, 3\}$ (at which point, we cannot remove 4 again). We can further remove 2 twice to obtain $\{1, 1|2, 2, 2, 3, 3\}$. Consequently, $\{4|\}$ and $\{1, 1|2, 3, 3\}$ as well as $\{2, 2, 4|\}$ and $\{1, 1|2, 2, 2, 3, 3\}$ are subsets of $\{1, 1, 4|2, 3, 3\}$.

Following [Loe92], a *new set* is a hybrid set such that either all multiplicities are 0 or 1 (a "positive set") or all multiplicities are 0 or -1 (a "negative set").

Theorem 4.3 ([Loe92]). *For all integers n and k, the number of k-element subsets of an n-element new set is $\left| \binom{n}{k} \right|$.*

Example 4.4. Consider the new set $\{|-1, -2, -3\}$ with -3 elements (the reason for choosing the elements to be negative numbers will become apparent when we revisit this example in Example 4.7). Its 2-element subsets are

$$\{-1, -1|\}, \quad \{-1, -2|\}, \quad \{-1, -3|\}, \quad \{-2, -2|\}, \quad \{-2, -3|\}, \quad \{-3, -3|\},$$

so that $\left| \binom{-3}{2} \right| = 6$. On the other hand, its -4-element subsets are

$$\{|-1, -1, -2, -3\}, \quad \{|-1, -2, -2, -3\}, \quad \{|-1, -2, -3, -3\},$$

so that $\left| \binom{-3}{-4} \right| = 3$.

Let X_n denote the standard new set with n elements, by which we mean $X_n = \{0, 1, \ldots, n-1|\}$, if $n \geq 0$, and $X_n = \{|-1, -2, \ldots, n\}$, if $n < 0$. For a hybrid set $Y \subseteq X_n$ with multiplicity function M_Y, we write

$$\sigma(Y) = \sum_{y \in Y} M_Y(y) y.$$

Note that, if Y is a classic set, then $\sigma(Y)$ is just the sum of its elements. With this setup, we prove the following uniform generalization of [Loe92, Theorem 5.2], which is well-known in the case that $n, k \geq 0$ (see, for instance, [KC02, Theorem 6.1]).

Theorem 4.5. *For all integers n and k,*

$$\binom{n}{k}_q = \varepsilon \sum_Y q^{\sigma(Y) - k(k-1)/2}, \quad \varepsilon = \pm 1, \tag{15}$$

where the sum is over all k-element subsets Y of the n-element set X_n. If $0 \leq k \leq n$, then $\varepsilon = 1$. If $n < 0 \leq k$, then $\varepsilon = (-1)^k$. If $k \leq n < 0$, then $\varepsilon = (-1)^{n-k}$.

Proof. The case $n, k \geq 0$ is well-known. A proof can be found, for instance, in [KC02, Theorem 6.1]. On the other hand, if $n \geq 0 > k$, then both sides vanish.

Let us consider the case $n < 0 \leq k$. It follows from the reflection formula (11) that (15) is equivalent to the (arguably cleaner, but less uniform because restricted to $n < 0 \leq k$) identity

$$\binom{k-n-1}{k}_q = \sum_{Y \in C(n,k)} q^{\sigma(Y)}, \tag{16}$$

where $C(n,k)$ is the collection of k-element subsets of the n-element set $X_n^+ = \{|0,1,2,\ldots,|n|-1\}$ (note that a natural bijection $X_n \to X_n^+$ is given by $x \mapsto |n| + x$).

Fix n, k and suppose that (16) holds whenever n and k are replaced with n' and k' such that $n < n' < 0$ or $n = n' < 0 \leq k' < k$. Then,

$$
\begin{aligned}
\sum_{Y \in C(n,k)} q^{\sigma(Y)} &= \sum_{\substack{Y \in C(n,k) \\ -n-1 \notin Y}} q^{\sigma(Y)} + \sum_{\substack{Y \in C(n,k) \\ -n-1 \in Y}} q^{\sigma(Y)} \\
&= \sum_{Y \in C(n+1,k)} q^{\sigma(Y)} + \sum_{Y \in C(n,k-1)} q^{\sigma(Y)-n-1} \\
&= \binom{k-n-2}{k}_q + q^{-n-1}\binom{k-n-2}{k-1}_q \\
&= \binom{k-n-1}{k}_q,
\end{aligned}
$$

where the last equality follows from Pascal's relation in the form (9). Since (16) holds trivially if $n = -1$ or if $k = 0$, it therefore follows by induction that (16) is true whenever $n < 0 \leq k$.

Finally, consider the case $n, k < 0$. It is clear that both sides vanish unless $k \leq n < 0$. By the third equality in Corollary 3.2,

$$\binom{n}{k}_q = (-1)^{n-k} q^{\frac{1}{2}(n(n+1)-k(k+1))}\binom{-k-1}{-n-1}_q,$$

so that (15) becomes equivalent to

$$\binom{-k-1}{-n-1}_q = \sum_{Y \in D(n,k)} q^{\sigma(Y)+k-n(n+1)/2}, \tag{17}$$

where $D(n,k)$ is the collection of k-element subsets Y of the n-element set $X_n = \{|-1,-2,\ldots,n\}$. If $n = -1$, then (17) holds because the only contribution comes from $Y = \{|-1,-1,\ldots,-1\}$, with $M_Y(-1) = |k|$ and $\sigma(Y) = -k$. If, on the other hand, $k = -1$, then (17) holds as well because a contributing Y only exists if $n = -1$. Fix $n, k < -1$ and suppose that (17) holds whenever n and k are replaced with n' and k' such that $k < k' < 0$ and $n \leq n' < 0$. Then the right-hand side of (17) equals

$$\sum_{\substack{Y \in D(n,k) \\ M_Y(n)=-1}} q^{\sigma(Y)+k-n(n+1)/2} + \sum_{\substack{Y \in D(n,k) \\ M_Y(n)<-1}} q^{\sigma(Y)+k-n(n+1)/2}.$$

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We now remove the element n from Y (once) and, to make up for that, replace $\sigma(Y)$ with $\sigma(Y)-n$. Proceeding thus, we see that the right-hand side of (17) equals

$$\sum_{Y \in D(n+1,k+1)} q^{\sigma(Y)+k+1-(n+1)(n+2)/2} + q^{-n-1} \sum_{Y \in D(n,k+1)} q^{\sigma(Y)+k+1-n(n+1)/2}$$

$$= \binom{-k-2}{-n-2}_q + q^{-n-1}\binom{-k-2}{-n-1}_q = \binom{-k-1}{-n-1}_q,$$

with the final equality following from Pascal's relation (7). We conclude, by induction, that (17) is true for all $n, k < 0$. $\qquad\square$

Remark 4.6. The number of possibilities to choose k elements from a set of n elements with replacement is

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}.$$

The usual "trick" to arrive at this count is to encode each choice of k elements by lining them up in some order with elements of the same kind separated by dividers (since there are n kinds of elements, we need $n-1$ dividers). The $n-1$ positions of the dividers among all $k+n-1$ positions then encode a choice of k elements. Formula (16) is a q-analog of this combinatorial fact.

Example 4.7. Let us revisit and refine Example 4.4, which concerns subsets of $X_{-3} = \{|-1, -2, -3\}$. Letting $k = 2$, the 2-element subsets have element-sums $\sigma(\{-1, -1|\}) = -2$, $\sigma(\{-1, -2|\}) = -3$, $\sigma(\{-1, -3|\}) = -4$, $\sigma(\{-2, -2|\}) = -4$, $\sigma(\{-2, -3|\}) = -5$, $\sigma(\{-3, -3|\}) = -6$. Subtracting $k(k-1)/2 = 1$ from these sums to obtain the weight, we find

$$\binom{-3}{2}_q = q^{-3} + q^{-4} + 2q^{-5} + q^{-6} + q^{-7}.$$

Next, let us consider the case $k = -4$. The -4-element subsets have element-sums

$$\sigma(\{|-1, -1, -2, -3\}) = 7, \quad \sigma(\{|-1, -2, -2, -3\}) = 8, \quad \sigma(\{|-1, -2, -3, -3\}) = 9.$$

Subtracting $k(k-1)/2 = 10$ from these sums and noting that $(-1)^{n-k} = -1$, we conclude that

$$\binom{-3}{-4}_q = -(q^{-3} + q^{-2} + q^{-1}).$$

In the remainder of this section, we consider two applications of Theorem 4.5. The first of these is the following extension of the classical Chu-Vandermonde identity.

Lemma 4.8. *For all integers n, m and k, with $k \geq 0$,*

$$\sum_{j=0}^{k} q^{(k-j)(n-j)} \binom{n}{j}_q \binom{m}{k-j}_q = \binom{n+m}{k}_q. \tag{18}$$

Proof. Throughout this proof, if Y is a k-element set, write $\tau(Y) = \sigma(Y) - k(k-1)/2$.

Suppose $n, m \geq 0$. Let Y_1 be a j-element subset of X_n, and Y_2 a $(k-j)$-element subset of X_m. Let $Y_2' = \{y + n : y \in Y_2\}$, so that $Y = Y_1 \cup Y_2'$ is a k-element subset of X_{n+m}. Then, since

$$\sigma(Y) = \sigma(Y_1) + \sigma(Y_2') = \sigma(Y_1) + \sigma(Y_2) + (k-j)n,$$

we have

$$\tau(Y) = \tau(Y_1) + \tau(Y_2) + (k-j)(n-j).$$

Then this follows from Theorem 4.5 because

$$\binom{j}{2} + \binom{k-j}{2} - \binom{k}{2} + (k-j)n = (k-j)(n-j).$$

$\square$

Similarly, one can deduce from Theorem 4.5 the following version for the case when k is a negative integer. It trivially also holds if $n, m \geq 0$, but the identity does not generally hold in the case when n and m have mixed signs.

Lemma 4.9. *For all negative integers n, m and k,*

$$\sum_{j \in \{-1, -2, \ldots, k+1\}} q^{(k-j)(n-j)} \binom{n}{j}_q \binom{m}{k-j}_q = \binom{n+m}{k}_q.$$

As another application of the combinatorial characterization in Theorem 4.5, we readily obtain the following identity. In the case $n \geq 0$, this identity is well-known and often referred to as the (commutative version of the) q-binomial theorem (in which case the sum only extends over $k = 0, 1, \ldots, n$). We will discuss the noncommutative q-binomial theorem in the next section.

Theorem 4.10. *For all integers n,*

$$(-x; q)_n = \sum_{k \geq 0} q^{k(k-1)/2} \binom{n}{k}_q x^k.$$

Proof. Suppose that $n \geq 0$, so that

$$(-x; q)_n = (1+x)(1+xq) \cdots (1 + xq^{n-1}). \tag{19}$$

Let, as before $X_n = \{0, 1, \ldots, n-1|\}$. To each subset $Y \subseteq X_n$ we associate the product of the terms xq^y with $y \in Y$ in the expansion of (19). This results in

$$(-x; q)_n = \sum_{Y \subseteq X_n} q^{\sigma(Y)} x^{|Y|},$$

which, by Theorem 4.5, reduces to the claimed sum.

Next, let us consider the case $n < 0$. Then, $X_n = \{|-1, -2, \ldots, n\}$ and

$$(x; q)_n = \prod_{j=1}^{|n|} \frac{1}{1 - xq^{-j}} = \prod_{j=1}^{|n|} \sum_{m \geq 0} x^m q^{-jm}.$$

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Similar to the previous case, terms of the expansion of this product are in natural correspondence with (hybrid) subsets $Y \subseteq X_n$. Namely, to Y we associate the product of the terms $x^m q^{ym}$ where $y \in Y$ and $m = M_Y(y)$ is the multiplicity of y. Therefore,

$$(-x; q)_n = \sum_{Y \subseteq X_n} (-1)^{|Y|} q^{\sigma(Y)} x^{|Y|},$$

and the claim again follows directly from Theorem 4.5 (note that $\varepsilon = (-1)^k$ in the present case). $\square$

5 The binomial theorem

When introducing binomial coefficients with negative entries, Loeb [Loe92] also provided an extension of the binomial theorem

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

the namesake of the binomial coefficients, to the case when n and k may be negative integers. In this section, we show that this extension can also be generalized to the case of q-binomial coefficients.

Suppose that $f(x)$ is a function with Laurent expansions

$$f(x) = \sum_{k \geq -N} a_k x^k, \qquad f(x) = \sum_{k \geq -N} b_{-k} x^{-k} \tag{20}$$

around $x = 0$ and $x = \infty$, respectively. Let us extract coefficients of these expansions by writing

$$\{x^k\} f(x) := \begin{cases} a_k, & \text{if } k \geq 0, \\ b_k, & \text{if } k < 0. \end{cases}$$

Loosely speaking, $\{x^k\} f(x)$ is the coefficient of x^k in the appropriate Laurent expansion of $f(x)$.

Theorem 5.1 ([Loe92]). *For all integers n and k,*

$$\binom{n}{k} = \{x^k\}(1 + x)^n.$$

Example 5.2. As $x \to \infty$,

$$(1 + x)^{-3} = x^{-3} - 3x^{-4} + 6x^{-5} + O(x^{-6}),$$

so that, for instance,

$$\binom{-3}{-5} = 6.$$

It is well-known (see, for instance, [KC02, Theorem 5.1]) that, if x and y are noncommuting variables such that $yx = qxy$, then the q-binomial coefficients arise from the expansion of $(x + y)^n$.

Theorem 5.3. *Let $n \geq 0$. If $yx = qxy$, then*

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}. \tag{21}$$

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Our next result shows that the restriction to $n \geq 0$ is not necessary. In fact, we prove the following result, which extends both the noncommutative q-binomial Theorem 5.3 and Loeb's Theorem 5.1. In analogy with the classical case, we consider expansions of $f_n(x, y) = (x + y)^n$ in the two q-commuting variables x, y. As before, we can expand $f_n(x, y)$ in two different ways, that is,

$$f_n(x, y) = \sum_{k \geq 0} a_k x^k y^{n-k}, \qquad f_n(x, y) = \sum_{k \geq n} b_{-k} x^{-k} y^{n+k}.$$

Again, we extract coefficients of these expansions by writing

$$\{x^k y^{n-k}\} f_n(x, y) := \begin{cases} a_k, & \text{if } k \geq 0, \\ b_k, & \text{if } k < 0. \end{cases}$$

Theorem 5.4. *Suppose that* $yx = qxy$. *Then, for all integers* n *and* k,

$$\binom{n}{k}_q = \{x^k y^{n-k}\}(x + y)^n.$$

Proof. Using the geometric series,

$$(x + y)^{-1} = y^{-1}(xy^{-1} + 1)^{-1} = y^{-1} \sum_{k \geq 0} (-1)^k (xy^{-1})^k.$$

and, applying the q-commutativity,

$$(x + y)^{-1} = \sum_{k \geq 0} (-1)^k q^{-k(k+1)/2} x^k y^{-k-1} = \sum_{k \geq 0} \binom{-1}{k}_q x^k y^{-1-k}.$$

(Consequently, the claim holds when $n = -1$ and $k \geq 0$.) More generally, we wish to show that, for all $n \geq 1$,

$$(x + y)^{-n} = \sum_{k \geq 0} \binom{-n}{k}_q x^k y^{-n-k}. \tag{22}$$

We just found that (22) holds for $n = 1$. On the other hand, assume that (22) holds for some n. Then,

$$
\begin{aligned}
(x + y)^{-n-1} &= (x + y)^{-n}(x + y)^{-1} \\
&= \left(\sum_{k \geq 0} \binom{-n}{k}_q x^k y^{-n-k} \right) \left(\sum_{k \geq 0} \binom{-1}{k}_q x^k y^{-1-k} \right) \\
&= \sum_{k \geq 0} \sum_{j=0}^{k} \binom{-n}{j}_q \binom{-1}{k-j}_q q^{(k-j)(-n-j)} x^k y^{-n-1-k} \\
&= \sum_{k \geq 0} \binom{-n-1}{k}_q x^k y^{-n-1-k},
\end{aligned}
$$

where the last step is an application of the generalized Chu-Vandermonde identity (18) with $m = -1$. By induction, (22) therefore is true for all $n \geq 1$.

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We have therefore shown that (21) holds for all integers n. This implies the present claim in the case $k \geq 0$. The case when $k < 0$ can also be deduced from (22). Indeed, observe that $xy = q^{-1}yx$, so that, for any integer n, by (21) and (22),

$$
\begin{aligned}
(x+y)^n &= \sum_{k \geq 0} \binom{n}{k}_{q^{-1}} y^k x^{n-k} \\
&= \sum_{k \leq n} q^{k(n-k)} \binom{n}{k}_{q^{-1}} x^k y^{n-k} = \sum_{k \leq n} \binom{n}{k}_q x^k y^{n-k}.
\end{aligned}
$$

When $n \geq 0$, this is just a version of (21). However, when $k < 0$, we deduce that

$$
\{x^k y^{n-k}\}(x+y)^n = \binom{n}{k}_q,
$$

as claimed. $\qquad \square$

6 Lucas' theorem

Lucas' famous theorem [Luc78] states that, if p is a prime, then

$$
\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1} \cdots \binom{n_d}{k_d} \pmod{p},
$$

where n_i and k_i are the p-adic digits of the nonnegative integers n and k, respectively. Our first goal is to prove that this congruence in fact holds for all integers n and k. The next section is then concerned with further extending these congruences to the polynomial setting.

Example 6.1. The base p expansion of a negative integer is infinite. However, only finitely many digits are different from $p - 1$. For instance, in base 7,

$$
-11 = 3 + 5 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \ldots
$$

which we will abbreviate as $-11 = (3, 5, 6, 6, \ldots)_7$. Similarly, $-19 = (2, 4, 6, 6, \ldots)_7$. The extension of the Lucas congruences that is proved below shows that

$$
\binom{-11}{-19} \equiv \binom{3}{2}\binom{5}{4}\binom{6}{6}\binom{6}{6} \cdots = 3 \cdot 5 \equiv 1 \pmod{7},
$$

without computing that the left-hand side is $43, 758$.

The main result of this section, Theorem 6.2, can also be deduced from the polynomial generalization in the next section. However, we give a direct and uniform proof here to make the ingredients more transparent. A crucial ingredient in the usual proofs of Lucas' classical theorem is the simple congruence

$$
(1 + x)^p \equiv 1 + x^p \pmod{p}, \tag{23}
$$

sometimes jokingly called a freshman's dream, which encodes the observation that $\binom{p}{k}$ is divisible by the prime p, except in the boundary cases $k = 0$ and $k = p$.

Theorem 6.2. *Let p be a prime. Then, for any integers n and k,*

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n'}{k'} \pmod{p},$$

where $n = n_0 + n'p$ and $k = k_0 + k'p$ with $n_0, k_0 \in \{0, 1, \ldots, p-1\}$.

Proof. It is a consequence of (23) (and the algebra of Laurent series) that, for any prime p,

$$(1+x)^{-p} \equiv (1+x^p)^{-1} \pmod{p}, \qquad (24)$$

where it is understood that both sides are expanded, as in (20), either around 0 or ∞. Hence, in the same sense,

$$(1+x)^{np} \equiv (1+x^p)^n \pmod{p} \qquad (25)$$

for any integer n.

With the notation from the previous section, we observe that

$$\{x^k\}(1+x)^n = \{x^k\}(1+x)^{n_0}(1+x)^{n'p} \equiv \{x^k\}(1+x)^{n_0}(1+x^p)^{n'} \pmod{p},$$

where the congruence is a consequence of (25). Since $n_0 \in \{0, 1, \ldots, p-1\}$, we conclude that

$$\{x^k\}(1+x)^n \equiv (\{x^{k_0}\}(1+x)^{n_0})(\{x^{k'p}\}(1+x^p)^{n'}) \pmod{p}.$$

This is obvious if $k \geq 0$, but remains true for negative k as well (because $(1+x)^{n_0}$ is a polynomial, in which case the expansions (20) around 0 and ∞ agree). Thus,

$$\{x^k\}(1+x)^n \equiv (\{x^{k_0}\}(1+x)^{n_0})(\{x^{k'}\}(1+x)^{n'}) \pmod{p}.$$

Applying Theorem 5.1 to each term, it follows that

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n'}{k'} \pmod{p},$$

as claimed. $\qquad\square$

7   A q-analog of Lucas' theorem

Let $\Phi_m(q)$ be the mth cyclotomic polynomial. In this section, we prove congruences of the type $A(q) \equiv B(q)$ modulo $\Phi_m(q)$, where $A(q), B(q)$ are Laurent polynomials. The congruence is to be interpreted in the natural sense that the difference $A(q) - B(q)$ is divisible by $\Phi_m(q)$.

Example 7.1. Following the notation in Theorem 6.2, in the case $(n, k) = (-4, -8)$, we have $(n_0, k_0) = (2, 1)$ and $(n', k') = (-2, -3)$. We reduce modulo $\Phi_3(q) = 1 + q + q^2$. The result we prove below shows that

$$\binom{-4}{-8}_q \equiv \binom{2}{1}_q \binom{-2}{-3} \pmod{\Phi_3(q)}.$$

Here,

$$\binom{-4}{-8}_q = \frac{1}{q^{22}}\Phi_5(q)\Phi_6(q)\Phi_7(q)$$

$$= \frac{1}{q^{22}}(1 - q + q^2)(1 + q + q^2 + q^3 + q^4)(1 + q + q^2 + \ldots + q^6)$$

as well as

$$\binom{2}{1}_q\binom{-2}{-3} = -2(1 + q),$$

and the meaning of the congruence is that

$$\binom{-4}{-8}_q - \binom{2}{1}_q\binom{-2}{-3} = \Phi_3(q) \cdot \frac{p_{21}(q)}{q^{22}},$$

where $p_{21}(q) = 1 + q^2 + 2q^3 + q^4 + \ldots - 2q^{19} + 2q^{21}$ is a polynomial of degree 21. Observe how, upon setting $q = 1$, we obtain the Lucas congruence

$$\binom{-4}{-8} \equiv \binom{2}{1}\binom{-2}{-3} \pmod{3},$$

provided by Theorem 6.2 (the two sides of the congruence are equal to 35 and -4, respectively).

In the case $n, k \geq 0$, the following q-analog of Lucas' classical binomial congruence has been obtained by Olive [Oli65] and Désarménien [Dés82]. A nice proof based on a group action is given by Sagan [Sag92], who attributes the combinatorial idea to Strehl. We show that these congruences extend uniformly to all integers n and k. A minor difference to keep in mind is that the q-binomial coefficients in this extended setting are Laurent polynomials (see Example 7.1).

Theorem 7.2. *Let $m \geq 2$ be an integer. For any integers n and k,*

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q\binom{n'}{k'} \pmod{\Phi_m(q)},$$

where $n = n_0 + n'm$ and $k = k_0 + k'm$ with $n_0, k_0 \in \{0, 1, \ldots, m - 1\}$.

Proof. Suppose throughout that x and y satisfy $yx = qxy$. It follows from the (noncommutative) q-binomial Theorem 5.3 that, for nonnegative integers m,

$$(x + y)^m \equiv x^m + y^m \pmod{\Phi_m(q)}.$$

As in the proof of Theorem 6.2 (and in the analogous sense), we conclude that

$$(x + y)^{nm} \equiv (x^m + y^m)^n \pmod{\Phi_m(q)} \tag{26}$$

for any integer n.

With the notation from Section 5, we observe that, by (26),

$$\{x^k y^{n-k}\}(x + y)^n \equiv \{x^k y^{n-k}\}(x + y)^{n_0}(x^m + y^m)^{n'} \pmod{\Phi_m(q)}.$$

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Since $n_0 \in \{0, 1, \ldots, p-1\}$, the right-hand side equals

$$q^{(n_0-k_0)k'm}(\{x^{k_0}y^{n_0-k_0}\}(x+y)^{n_0})(\{x^{k'm}y^{(n'-k')m}\}(x^m+y^m)^{n'}).$$

As $q^m \equiv 1$ modulo $\Phi_m(q)$, we conclude that $\{x^k y^{n-k}\}(x+y)^n$ is congruent to

$$(\{x^{k_0}y^{n_0-k_0}\}(x+y)^{n_0})(\{x^{k'm}y^{(n'-k')m}\}(x^m+y^m)^{n'})$$

modulo $\Phi_m(q)$. Observe that the variables $X = x^m$ and $Y = y^m$ satisfy the commutation relation $YX = q^{m^2}XY$. Hence, applying Theorem 5.4 to each term, we conclude that

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'}_{q^{m^2}} \pmod{\Phi_m(q)}.$$

Since $q^{m^2} \equiv 1$ modulo $\Phi_m(q)$, the claim follows. $\qquad\square$

In [ABDJ17], Adamczewski, Bell, Delaygue and Jouhet consider congruences modulo cyclotomic polynomials for multidimensional q-factorial ratios and are thus able to generalize many Lucas-type congruences. In particular, specializing [ABDJ17, Proposition 1.4] (the case $q = 1$ of which had previously been proved in [ABD16]) to $d = 2$, $u = 1$, $v = 2$, $e_1 = (1; 0)$, $f_1 = (1; -1)$ and $f_2 = (0; 1)$, we obtain the classical case $n, k \geq 0$ of Theorem 7.2. As pointed out by Adamczewski, Bell, Delaygue and Jouhet in private communication, an alternative, a little more tricky, proof of the general case of Theorem 7.2 can be obtained by reducing it, via Corollary 3.2, to the nonnegative case.

8 Conclusion

We believe (and hope that the results of this paper provide some evidence to that effect) that the binomial and q-binomial coefficients with negative entries are natural and beautiful objects. On the other hand, let us indicate an application, taken from [Str14], of binomial coefficients with negative entries.

Example 8.1. A crucial ingredient in Apéry's proof [Apé79] of the irrationality of $\zeta(3)$ is played by the *Apéry numbers*

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2. \tag{27}$$

These numbers have many interesting properties. For instance, they satisfy remarkably strong congruences, including

$$A(p^r m - 1) \equiv A(p^{r-1}m - 1) \pmod{p^{3r}}, \tag{28}$$

established by Beukers [Beu85], and

$$A(p^r m) \equiv A(p^{r-1}m) \pmod{p^{3r}}, \tag{29}$$

proved by Coster [Cos88]. Both congruences hold for all primes $p \geq 5$ and positive integers m, r. The definition of the Apéry numbers $A(n)$ can be extended to all integers n by setting

$$A(n) = \sum_{k \in \mathbb{Z}} \binom{n}{k}^2 \binom{n+k}{k}^2, \tag{30}$$

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where the binomial coefficients are now allowed to have negative entries. Applying the reflection rule (10) to (30), we obtain

$$A(-n) = A(n-1). \tag{31}$$

In particular, we find that the congruence (28) is equivalent to (29) with m replaced with $-m$. By working with binomial coefficients with negative entries, the second author gave a uniform proof of both sets of congruences in [Str14]. In addition, the symmetry (31), which becomes visible when allowing negative indices, explains why other Apéry-like numbers satisfy (29) but not (28).

We illustrated that the Gaussian binomial coefficients can be usefully extended to the case of negative arguments. More general binomial coefficients, formed from an arbitrary sequence of integers, are considered, for instance, in [KW89] and it is shown by Hu and Sun [HS01] that Lucas' theorem can be generalized to these. It would be interesting to investigate the extent to which these coefficients and their properties can be extended to the case of negative arguments. Similarly, an elliptic analog of the binomial coefficients has recently been introduced by Schlosser [Sch11], who further obtains a general noncommutative binomial theorem of which Theorem 5.3 is a special case. It is natural to wonder whether these binomial coefficients have a natural extension to negative arguments as well.

In the last section, we showed that the generalized q-binomial coefficients satisfy Lucas congruences in a uniform fashion. It would be of interest to determine whether other well-known congruences for the q-binomial coefficients, such as those considered in [And99] or [Str11], have similarly uniform extensions.

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