Supercongruences for polynomial analogs of the Apéry numbers

Armin Straub*
Department of Mathematics and Statistics
University of South Alabama

March 19, 2018

Abstract

We consider a family of polynomial analogs of the Apéry numbers, which includes q-analogs of Krattenthaler–Rivoal–Zudilin and Zheng, and show that the supercongruences that Gessel and Mimura established for the Apéry numbers generalize to these polynomials. Our proof relies on polynomial analogs of classical binomial congruences of Wolstenholme and Ljunggren. We further indicate that this approach generalizes to other supercongruence results.

1 Introduction

Among the many interesting properties of the *Apéry numbers*

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \tag{1}$$

which are at the heart of R. Apéry's proof [7], [31] of the irrationality of $\zeta(3)$, are congruences with surprisingly large modulus. Following F. Beukers [8] these are often referred to as *supercongruences*. For instance, for all primes $p \geq 5$,

$$A(pn) \equiv A(n) \pmod{p^3}, \tag{2}$$

as conjectured by S. Chowla, J. Cowles and M. Cowles [12] and proved by I. Gessel [19] and Y. Mimura [26]. When n is divisible by p then this congruence can be further strengthened [8], [15]. Indeed, the congruence $A(p^r n) \equiv A(p^{r-1} n)$ holds modulo p^{3r}.

In this paper, we are concerned with polynomial analogs of the Apéry numbers (1) and it is our goal to demonstrate that these polynomials share some of

* *Email:* `straub@southalabama.edu`

the remarkable arithmetic properties. In Section 3, we review q-analogs of the Apéry numbers featuring in work of C. Krattenthaler, T. Rivoal and W. Zudilin [24] and D. Zheng [35], as well as other natural constructions. In particular, in Lemma 3.1, we show that the q-numbers of Krattenthaler, Rivoal and Zudilin, which are defined via a q-partial fraction decomposition, essentially have the explicit q-binomial representation

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

closely resembling (1).

In Section 4, we prove that the polynomials $A_q(n)$ satisfy congruences modulo cubes of the mth cyclotomic polynomials $\Phi_m(q)$. When specialized to $q = 1$ and $m = p$, these congruences imply the known supercongruences (2).

Corollary 1.1. *For any integer m,*

$$A_q(mn) \equiv A_{q^{m^2}}(n) - \frac{m^2-1}{12}(q^m - 1)^2 n^2 A_1(n) \pmod{\Phi_m(q)^3}. \tag{3}$$

This result is a consequence of Theorem 4.1, our main theorem, which offers a more general multivariate version. To the best of our knowledge, congruence (3) is the first polynomial analog of a supercongruence of the type (2). The congruences (2) are conjectured to hold for all Apéry-like sequences (see [29] as well as Section 5) but remain open in some cases. It is our hope that understanding these congruences in the most general setting might help shed light on the more mysterious cases. Another reason to be interested in polynomial analogs is the availability of additional techniques not available outside the q-world, illustrated, for instance, in the very recent work [22] of V. Guo and W. Zudilin. In particular, for supercongruences of a different type, Guo and Zudilin succeed in introducing an additional variable a (so that the limit $a \to 1$ recovers the original congruence) in such a way that the generalized congruence holds modulo $(1-q^n)(a-q^n)(1-aq^n)/(1-q)$. The crucial benefit is that this generalized congruence can be established modulo $(1-q^n)/(1-q)$, $a - q^n$ and $1 - aq^n$ individually, since these polynomials are coprime. These three individual congruences then correspond to (exactly) evaluating the terms in question when q is an nth root of unity (for $(1-q^n)/(1-q)$) and when $a = q^{\pm n}$ (for $a - q^n$ and $1 - aq^n$). It would be of considerable interest to determine whether the congruences considered herein could be similarly approached (and, to some extent, better explained) by (creatively!) introducing an appropriate additional variable a (this is referred to as *creative microscoping* in [22]).

Supercongruence results for other sequences, that are based on a suitable binomial sum representation and analogous arguments, can be generalized likewise to the polynomial setting. We illustrate that point at the example of the family of generalized Apéry sequences

$$A^{(\lambda,\mu)}(n) = \sum_{k=0}^{n} \binom{n}{k}^{\lambda} \binom{n+k}{k}^{\mu}. \tag{4}$$

E. Deutsch and B. Sagan [17] showed that, if $\lambda \geq 2$ and $\mu \geq 1$, these sequences satisfy the supercongruences

$$A^{(\lambda,\mu)}(pn) \equiv A^{(\lambda,\mu)}(n) \pmod{p^3} \tag{5}$$

for all primes $p \geq 5$ (in fact, a generalized version of these congruences already appears in M. Coster's thesis [15]). The congruences (2) are the special case $(\lambda, \mu) = (2, 2)$.

We prove the following polynomial analog of the congruences (5). Observe how the case $(\lambda, \mu) = (2, 2)$ reduces to the Apéry number case of Corollary 1.1, in which case the congruences take a particularly clean form.

Theorem 1.2. *For fixed integers λ, μ such that $\lambda \geq 2$ and $\mu \geq 0$, as well as a polynomial $\alpha \in \mathbb{Z}[n, k]$ such that $\alpha(mn, mk) = m^2\alpha(n, k)$ and $\alpha(0, k) = k^2$, define*

$$A_q^{(\lambda,\mu)}(n) = \sum_{k=0}^{n} q^{\alpha(n,k)} \binom{n}{k}_q^{\lambda} \binom{n+k}{k}_q^{\mu}. \tag{6}$$

Then, for any positive integer m,

$$A_q^{(\lambda,\mu)}(mn) \equiv A_{q^{m^2}}^{(\lambda,\mu)}(n) - \frac{m^2-1}{12}(q^m-1)^2 R^{(\lambda,\mu)}(n) \pmod{\Phi_m(q)^3}, \tag{7}$$

where the numbers

$$R^{(\lambda,\mu)}(n) = \sum_{k\geq 0} c_{n,k}^{(\lambda,\mu)} \binom{n}{k}^{\lambda} \binom{n+k}{k}^{\mu}, \quad c_{n,k}^{(\lambda,\mu)} = \begin{cases} n^2 + (\mu-2)\frac{nk}{2}, & \text{if } \lambda = 2, \\ ((\lambda+\mu)n - \lambda k)\frac{k}{2}, & \text{if } \lambda > 2, \end{cases}$$

are independent of α and m.

Our proof of this result proceeds analogously to the proof of Theorem 4.1 and is therefore omitted.

Example 1.3. Note that, specializing Theorem 1.2 to $(\lambda, \mu) = (2, 0)$ and $\alpha(n, k) = k^2$, we obtain the sequence

$$\sum_{k=0}^{n} q^{k^2} \binom{n}{k}_q^2 = \binom{2n}{n}_q$$

of central q-binomial coefficients. Because of the simple identity

$$\sum_{k=0}^{n} (n-k) \binom{n}{k}^2 = \frac{n}{2}\binom{2n}{n},$$

the resulting congruences (7) simplify to

$$\binom{2mn}{mn}_q \equiv \binom{2n}{n}_{q^{m^2}} - \frac{m^2-1}{12}(q^m-1)^2 \frac{n^2}{2}\binom{2n}{n} \pmod{\Phi_m(q)^3}.$$

This matches the special case $a = 2m$, $b = m$ of (15). In the case $n = 1$ this further reduces to (37), which is proved in the appendix and is a crucial ingredient for the other polynomial congruences proved herein.

2 Notations and conventions

Throughout, all congruences, such as (3), modulo a polynomial $\varphi(q)$ are understood in the ring $\mathbb{Q}[q^{\pm 1}]$ of Laurent polynomials with rational coefficients. In other words,

$$f(q) \equiv 0 \pmod{\varphi(q)} \tag{8}$$

means that $f(q) = g(q)\varphi(q)$ for some Laurent polynomial $g(q) \in \mathbb{Q}[q^{\pm 1}]$. In all congruences considered herein, $\varphi(q)$ is a monic polynomial with integer coefficients. On the other hand, we allow $f(q)$ to have rational coefficients. For instance, the coefficients of $f(q)$ typically involve quantities like $(m^2-1)/12$ which are not integral if $\gcd(m, 6) > 1$. Observe that, as a consequence of Gauss' lemma, if $\varphi(q)$ and $f(q)$ both have integer coefficients and $\varphi(q)$ is monic, then $f(q) = g(q)\varphi(q)$ implies that $g(q)$ has integer coefficients as well. Therefore, in that case, the congruence (8) holds in the ring $\mathbb{Z}[q^{\pm 1}]$, so that, in particular, we also have the ordinary congruence

$$f(1) \equiv 0 \pmod{\varphi(1)}. \tag{9}$$

Since our congruences are modulo powers of cyclotomic polynomials $\Phi_m(q)$, it is useful to recall that $\Phi_m(1) = p$ if m is a power of the prime p. However, $\Phi_m(1) = 1$ if m is not a prime power, in which case the congruence (9) carries no information (in contrast to the polynomial congruence (8)).

As usual, the q-analog of an integers $n \geq 0$ is the polynomial $[n]_q = 1 + q + \cdots + q^{n-1}$. Thus equipped, one introduces the q-factorial as $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ and the q-binomial coefficient as

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{10}$$

It is easy to see from here that the q-binomial coefficient is a self-reciprocal polynomial in q of degree $k(n-k)$. The q-binomial coefficient occurs naturally in many contexts [23]; for instance, if q is a prime power, then (10) counts the number of k-dimensional subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$. For our purposes, the following alternative characterization will also be relevant. Suppose that x and y are q-commuting in the sense that

$$yx = qxy \tag{11}$$

(of course, in the case $q = 1$, the variables x and y indeed commute). Then we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \tag{12}$$

extending the classical binomial expansion.

The following q-analog of Lucas' classical binomial congruence is proved by G. Olive [27] and J. Désarménien [16]. More recently, it is shown in [18] that this result extends to the case when $na + b$ and $nr + s$ are allowed to be negative integers. In [1], B. Adamczewski, J. Bell, É. Delaygue and F. Jouhet consider congruences modulo cyclotomic polynomials for multidimensional q-factorial ratios, which generalize this and many other Lucas-type congruences.

Lemma 2.1. *For any integers $n, a, b, r, s \geq 0$ such that $b, s < n$,*

$$\binom{an+b}{rn+s}_q \equiv \binom{a}{r}\binom{b}{s}_q \pmod{\Phi_n(q)}. \tag{13}$$

We next establish a polynomial analog of the classical congruence

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}, \tag{14}$$

valid for primes $p \geq 5$. Congruence (14) is an extension of Wolstenholme's congruence (which corresponds to $a = 2$ and $b = 1$) and was proved in 1952 by Ljunggren, see [21]. A q-analog of (14) modulo p^2 was proved by G. Andrews [6], who asked for an extension modulo p^3. The next congruence is such an extension.

Theorem 2.2. *For positive integers n and nonnegative integers a, b,*

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}. \tag{15}$$

This result is proved in [33] in the case when n is prime. Note that, by the comments after (8), setting $q = 1$ in congruence (15) indeed recovers (14) for primes $p \geq 5$. Our proof of Theorem 2.2 is a straightforward extension of the corresponding proof given in [33], where congruence (15) is proved in the case that n is a prime. We include the details of the proof in Appendix A for the benefit of the reader.

3 q-analogs of the Apéry numbers

In their investigation of irrationality results on q-analogs of Riemann zeta values, C. Krattenthaler, T. Rivoal and W. Zudilin [24] implicitly introduce a q-analog $A_q^{\mathrm{KRZ}}(n)$ of the Apéry numbers as follows. In order to obtain irrationality results on

$$\zeta_q(3) = \sum_{k=1}^{\infty} \frac{q^k(1+q^k)}{(1-q^k)^3},$$

a q-analog of $\zeta(3)$, they consider linear forms in 1 and $\zeta_q(3)$. The coefficients of these linear forms are Laurent polynomials in q, and $A_q^{\mathrm{KRZ}}(n)$ is the coefficient of $\zeta_q(3)$. Specifically, the Laurent polynomials $A_q^{\mathrm{KRZ}}(n)$ and $B_q^{\mathrm{KRZ}}(n)$ are characterized by

$$\frac{1}{\log q}\sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{(q^{k-n};q)_n^2}{(q^k;q)_{n+1}^2} q^k\right) = A_q^{\mathrm{KRZ}}(n)\zeta_q(3) - B_q^{\mathrm{KRZ}}(n). \tag{16}$$

The (Laurent) polynomials $A_q^{\mathrm{KRZ}}(n)$ have the following explicit formula as a q-binomial sum which visibly reduces to Apéry's binomial sum (1) when $q = 1$.

Lemma 3.1. *With $A_q^{\mathrm{KRZ}}(n)$ defined as in (16),*

$$q^{n(2n+1)} A_q^{\mathrm{KRZ}}(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2. \tag{17}$$

Proof. It is shown in [24] that

$$A_q^{\mathrm{KRZ}}(n) = \sum_{k=0}^{n} \frac{a_q(n,k)}{q^k}, \tag{18}$$

where $a_q(n,k)$ is defined via the q-partial fraction decomposition

$$\frac{(q^{-n}T;q)_n^2}{(T;q)_{n+1}^2} = \sum_{j=0}^{n} \left(\frac{a_q(n,k)}{(1-q^kT)^2} + \frac{b_q(n,k)}{1-q^kT} \right),$$

where, as usual, $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the q-Pochhammer symbol. Therefore, by construction,

$$a_q(n,k) = \lim_{T\to q^{-k}} (1-q^kT)^2 \frac{(q^{-n}T;q)_n^2}{(T;q)_{n+1}^2} = \left[\lim_{T\to q^{-k}} (1-q^kT) \frac{(q^{-n}T;q)_n}{(T;q)_{n+1}} \right]^2.$$

Expanding the q-Pochhammer symbols, canceling the factor $1-q^kT$, and setting $T = q^{-k}$ in the resulting expression, we find that

$$\lim_{T\to q^{-k}} (1-q^kT) \frac{(q^{-n}T;q)_n}{(T;q)_{n+1}} = \frac{(q^{-1};q^{-1})_{n+k}}{(q^{-1};q^{-1})_k^2 (q;q)_{n-k}}.$$

Using the transformation formula

$$(q;q)_n = (-1)^n q^{\binom{n+1}{2}} (q^{-1};q^{-1})_n,$$

we observe that

$$\frac{(q^{-1};q^{-1})_{n+k}}{(q^{-1};q^{-1})_k^2 (q;q)_{n-k}} = (-1)^{n+k} q^{2\binom{k+1}{2} - \binom{n+k+1}{2}} \binom{n}{k}_q \binom{n+k}{k}_q.$$

We hence conclude that

$$a_q(n,k) = q^{2k(k+1)-(n+k)(n+k+1)} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

which, in combination with (18), implies (17). □

Investigating q-analogs of a result of S. Ahlgren and K. Ono [3] and W. Chu [13] pertaining to a modular supercongruence that was conjectured by F. Beukers [9] (see Section 5 and (34) for further details), D. Zheng [35] introduces the q-Apéry numbers

$$\sum_{k=0}^{n} q^{k(k-2n)} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2 = q^{n(n+1)} A_q^{\mathrm{KRZ}}(n). \tag{19}$$

and obtains the identity

$$\sum_{k=0}^{n} q^{k(k-2n)} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2 \{2H_q(k) - H_q(n+k) - qH_{1/q}(n-k)\} = 0, \tag{20}$$

involving the q-harmonic numbers

$$H_q(n) = \sum_{k=1}^{n} \frac{1}{[k]_q}.$$

In the remainder of this section, we illustrate that the q-Apéry numbers considered by Krattenthaler–Rivoal–Zudilin and Zheng can also be obtained as special cases of a natural construction based on introducing q-commuting variables.

It was shown in [34] that the Apéry numbers are the diagonal Taylor coefficients of the rational function

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4) - x_1x_2x_3x_4} = \sum_{n_1,n_2,n_3,n_4 \geq 0} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}. \tag{21}$$

Here, $\boldsymbol{n} = (n_1, \ldots, n_4)$ and $\boldsymbol{x}^{\boldsymbol{n}} = x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}$. The multivariate Apéry numbers $A(\boldsymbol{n})$ featuring in this expansion have the explicit representation

$$A(\boldsymbol{n}) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k}\binom{n_3}{k}\binom{n_1+n_2-k}{n_1}\binom{n_3+n_4-k}{n_3}. \tag{22}$$

Example 3.2. In search for a natural q-analog of these multivariate Apéry numbers, it is reasonable to consider sums of q-binomial coefficients of the form

$$\sum_{k \geq 0} q^{h(\boldsymbol{n};k)} \binom{n_1}{k}_q \binom{n_3}{k}_q \binom{n_1+n_2-k}{n_1}_q \binom{n_3+n_4-k}{n_3}_q,$$

where $h(\boldsymbol{n};k)$ is a polynomial in $\boldsymbol{n}$ and k, taking integer values. We note that a particularly natural choice is $h(\boldsymbol{n};k) = k^2$ because then the resulting q-numbers are self-reciprocal polynomials in q of degree $n_1n_2 + n_3n_4$. To see this, observe that

$$\deg \binom{n_1}{k}_q \binom{n_3}{k}_q \binom{n_1+n_2-k}{n_1}_q \binom{n_3+n_4-k}{n_3}_q = n_1n_2 + n_3n_4 - 2k^2.$$

Consequently, since each q-binomial is self-reciprocal, q^{k^2} times this product of four q-binomials is a polynomial $z(q)$ satisfying $z(q) = q^{n_1n_2+n_3n_4} z(1/q)$.

A more organic approach to obtaining the q-analog in Example 3.2, as well as other variations, is based on the observation made in [34] that, by MacMahon's Master Theorem [25], the power series expansion (21) defining the multivariate Apéry numbers $A(\boldsymbol{n})$ is equivalent to

$$A(\boldsymbol{n}) = [\boldsymbol{x}^{\boldsymbol{n}}](x_1 + x_2 + x_3)^{n_1}(x_1 + x_2)^{n_2}(x_3 + x_4)^{n_3}(x_2 + x_3 + x_4)^{n_4},$$

where $A(\boldsymbol{n})$ is represented as the coefficient of $\boldsymbol{x}^{\boldsymbol{n}}$ in a certain polynomial.

In light of the q-binomial expansion (12), it is natural to define a q-analog of $A(\boldsymbol{n})$ by extracting the coefficient of $\boldsymbol{x}^{\boldsymbol{n}}$ in the product of $(x_1 + x_2 + x_3)^{n_1}$, $(x_1 + x_2)^{n_2}$, $(x_3 + x_4)^{n_3}$ and $(x_2 + x_3 + x_4)^{n_4}$, where now the variables x_i are assumed to be q-commuting in the spirit of (11). Depending on the exact choices, including the order of the factors, one obtains different q-analogs.

Example 3.3. Suppose that, for all $i < j$, the variables x_i commute according to $x_j x_i = q x_i x_j$. Let $A_q(\boldsymbol{n})$ denote the coefficient of $\boldsymbol{x}^{\boldsymbol{n}}$ in the product

$$(x_1 + x_2 + x_3)^{n_1}(x_1 + x_2)^{n_2}(x_3 + x_4)^{n_3}(x_2 + x_3 + x_4)^{n_4}.$$

Then, using the q-binomial expansion (12), we obtain that

$$A_q(\boldsymbol{n}) = \sum_{k=0}^{\min(\boldsymbol{n})} q^{k(n_2+n_3+k)} \binom{n_1}{k}_q \binom{n_3}{k}_q \binom{n_1+n_2-k}{n_1}_q \binom{n_3+n_4-k}{n_4}_q.$$

Example 3.4. On the other hand, if we modify the previous example to let $A_q(\boldsymbol{n})$ denote the coefficient of $\boldsymbol{x}^{\boldsymbol{n}}$ in the product

$$(x_1 + x_2)^{n_2}(x_1 + x_2 + x_3)^{n_1}(x_2 + x_3 + x_4)^{n_4}(x_3 + x_4)^{n_3},$$

then we find that

$$A_q(\boldsymbol{n}) = \sum_{k=0}^{\min(\boldsymbol{n})} q^{k^2} \binom{n_1}{k}_q \binom{n_3}{k}_q \binom{n_1+n_2-k}{n_1}_q \binom{n_3+n_4-k}{n_4}_q,$$

which is the symmetric q-analog corresponding to the choice $h(\boldsymbol{n}; k) = k^2$ in Example 3.2. Specializing to $n_1 = n_2 = n_3 = n_4 = n$, the polynomial $A_q(\boldsymbol{n})$ equals (17), and we recover the q-Apéry numbers considered by Krattenthaler–Rivoal–Zudilin and Zheng.

In the next section, we will show that all of the q-analogs, including the multivariate ones from Examples 3.3 and 3.4, discussed in this section satisfy supercongruences generalizing the congruences (2).

4 Proof of the supercongruences

This section is concerned with proving the following multivariate generalization of Corollary 1.1. This generalization is inspired by the corresponding classical congruences, which were shown in [34] and which can be obtained from Theorem 4.1 by specializing $q = 1$.

Theorem 4.1. *For $\alpha \in \mathbb{Z}[\boldsymbol{n}, k]$ such that $\alpha(m\boldsymbol{n}, mk) = m^2\alpha(\boldsymbol{n}, k)$ and $\alpha(\boldsymbol{0}, k) = k^2$, define*

$$A_q^{(\alpha)}(\boldsymbol{n}) = \sum_{k\in\mathbb{Z}} q^{\alpha(\boldsymbol{n},k)} \binom{n_1}{k}_q \binom{n_3}{k}_q \binom{n_1+n_2-k}{n_1}_q \binom{n_3+n_4-k}{n_3}_q. \tag{23}$$

Then, for any positive integer m,

$$A_q^{(\alpha)}(m\boldsymbol{n}) \equiv A_{q^{m^2}}^{(\alpha)}(\boldsymbol{n}) - \frac{m^2-1}{12}(q^m-1)^2 R(\boldsymbol{n}) \pmod{\Phi_m(q)^3}, \tag{24}$$

where the numbers

$$R(\boldsymbol{n}) = \frac{n_1n_2 + n_3n_4}{2} A_1^{(\alpha)}(\boldsymbol{n})$$

are independent of α and m.

Proof. We have

$$A_q^{(\alpha)}(\boldsymbol{n}) = \sum_{k\geq 0} B_q(\boldsymbol{n}; k)$$

with

$$B_q(\boldsymbol{n}; k) = q^{\alpha(\boldsymbol{n},k)} \binom{n_1}{k}_q \binom{n_3}{k}_q \binom{n_1+n_2-k}{n_1}_q \binom{n_3+n_4-k}{n_3}_q.$$

Following the approach of [19], we write

$$A_q^{(\alpha)}(m\boldsymbol{n}) = S_q^{(1)}(\boldsymbol{n}) + S_q^{(2)}(\boldsymbol{n}), \tag{25}$$

where

$$S_q^{(1)}(\boldsymbol{n}) = \sum_{m|k} B_q(m\boldsymbol{n}; k), \quad S_q^{(2)}(\boldsymbol{n}) = \sum_{j=1}^{m-1} \sum_{k\geq 0} B_q(m\boldsymbol{n}; mk + j).$$

Applying Theorem 2.2, the q-analog of Ljunggren's binomial congruence, to each of the binomial coefficients in the summand of $S_q^{(1)}(\boldsymbol{n})$ and simplifying (keeping in mind that $q^m \equiv 1$ modulo $\Phi_m(q)$), we obtain that

$$S_q^{(1)}(\boldsymbol{n}) \equiv A_{q^{m^2}}^{(\alpha)}(\boldsymbol{n}) - \frac{m^2-1}{12}(q^m-1)^2 R^{(1)}(\boldsymbol{n}) \pmod{\Phi_m(q)^3}, \tag{26}$$

where

$$\begin{aligned} R^{(1)}(\boldsymbol{n}) &= \sum_{k\geq 0} \left[\frac{(n_1-k)k}{2} + \frac{(n_3-k)k}{2} + \frac{(n_2-k)n_1}{2} + \frac{(n_4-k)n_3}{2} \right] C(\boldsymbol{n}; k) \\ &= \sum_{k\geq 0} \left[\frac{n_1n_2 + n_3n_4}{2} - k^2 \right] C(\boldsymbol{n}; k) \end{aligned}$$

with
$$C(\boldsymbol{n};k)=\binom{n_1}{k}\binom{n_3}{k}\binom{n_1+n_2-k}{n_1}\binom{n_3+n_4-k}{n_3}. \tag{27}$$

Note that $q^m\equiv 1$ modulo $\Phi_m(q)$, which is why the term $q^{\alpha(m\boldsymbol{n},mk)}=q^{m^2\alpha(\boldsymbol{n},k)}$ in $B_q(m\boldsymbol{n};mk)$ does not contribute to $R^{(1)}(\boldsymbol{n})$.

It therefore remains to consider $S_q^{(2)}(\boldsymbol{n})$ modulo $\Phi_m(q)^3$. It is proved in Proposition 4.2 that

$$\binom{mn}{mk+j}_q\equiv(-1)^{j-1}q^{(j-1)(m-j/2)}\frac{[mn]_q}{[j]_q}\binom{n-1}{k}\pmod{\Phi_m(q)^2}.$$

On the other hand, it follows from Lemma 2.1, the q-analog of Lucas' congruence, that, for $0<j\le m$,

$$\binom{m(n+k)-j}{mn}_q=\binom{m(n+k-1)+(m-j)}{mn}_q\equiv\binom{n+k-1}{n}\pmod{\Phi_m(q)}.$$

Note that the congruence holds trivially in the case $n=k=0$.

Thus, modulo $\Phi_m(q)^3$,

$$\begin{aligned}
S_q^{(2)}(\boldsymbol{n}) &\equiv \sum_{j=1}^{m-1}\sum_{k\ge 0}q^{\alpha(m\boldsymbol{n},mk+j)}q^{(j-1)(2m-j)}\frac{[mn_1]_q[mn_3]_q}{[j]_q^2}\\
&\qquad\times\binom{n_1-1}{k}\binom{n_3-1}{k}\binom{n_1+n_2-k-1}{n_1}\binom{n_3+n_4-k-1}{n_3}\\
&\equiv \frac{[mn_1]_q[mn_3]_q}{n_1n_3}\sum_{j=1}^{m-1}\frac{q^j}{[j]_q^2}\sum_{k\ge 0}(k+1)^2C(\boldsymbol{n};k+1),
\end{aligned}\tag{28}$$

with $C(\boldsymbol{n};k)$ as in (27). For the second congruence, we used the presence of the term $[mn_1]_q[mn_3]_q$, which is divisible by $\Phi_m(q)^2$, together with

$$q^{\alpha(m\boldsymbol{n},mk+j)}q^{(j-1)(2m-j)}\equiv q^{\alpha(\boldsymbol{0},j)}q^{j-j^2}\equiv q^j\pmod{\Phi_m(q)},$$

to reduce the powers of q modulo $\Phi_m(q)^3$.

Note that it follows from the simple observation

$$\sum_{j=1}^{m-1}\frac{q^j}{[j]_q^2}=\sum_{j=1}^{m-1}\frac{1}{[j]_q^2}+(q-1)\sum_{j=1}^{m-1}\frac{1}{[j]_q},$$

combined with the congruences (39) and (40), that, for all integers m,

$$\sum_{j=1}^{m-1}\frac{q^j}{[j]_q^2}\equiv-\frac{m^2-1}{12}(q-1)^2\pmod{\Phi_m(q)}.$$

Further using that

$$(q-1)^2[mn_1]_q[mn_3]_q\equiv(q^m-1)^2n_1n_3\pmod{\Phi_m(q)^3},$$

we conclude from (28) that

$$S_q^{(2)}(\boldsymbol{n}) \equiv -\frac{m^2-1}{12}(q^m-1)^2 \sum_{k\geq 0} k^2 C(\boldsymbol{n};k) \pmod{\Phi_m(q)^3}. \tag{29}$$

Combining (25), (26) and (29), we therefore have

$$A_q^{(\alpha)}(m\boldsymbol{n}) \equiv A_{q^{m^2}}^{(\alpha)}(\boldsymbol{n}) - \frac{m^2-1}{12}(q^m-1)^2 \frac{n_1 n_2 + n_3 n_4}{2} \sum_{k\geq 0} C(\boldsymbol{n};k) \pmod{\Phi_m(q)^3},$$

which is the claimed congruence (24). □

Proposition 4.2. *For nonnegative integers j,k,m,n, with $0<j<m$,*

$$\binom{mn}{mk+j}_q \equiv (-1)^{j-1} q^{(j-1)(m-j/2)} \frac{[mn]_q}{[j]_q} \binom{n-1}{k} \pmod{\Phi_m(q)^2}.$$

Proof. We have

$$\begin{aligned} \binom{mn}{mk+j}_q &= \frac{[mn]_q}{[mk+j]_q} \binom{mn-1}{mk+j-1}_q \\ &\equiv \frac{[mn]_q}{[j]_q} \binom{mn-1}{mk+j-1}_q \pmod{\Phi_m(q)^2}, \end{aligned}$$

because $[ab]_q = [a]_{q^b}[b]_q$ and $[a+b]_q = [a]_q + q^a [b]_q$. Moreover, using Lemma 2.1, the q-analog of Lucas' binomial congruence, if $0<j<m$,

$$\begin{aligned} \binom{mn-1}{mk+j-1}_q &= \binom{m(n-1)+m-1}{mk+j-1}_q \\ &\equiv \binom{n-1}{k}\binom{m-1}{j-1}_q \pmod{\Phi_m(q)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \binom{m-1}{j-1}_q &= \prod_{k=1}^{j-1} \frac{[m-k]_q}{[k]_q} = \prod_{k=1}^{j-1} \frac{[m]_q - q^{m-k}[k]_q}{[k]_q} \\ &\equiv (-1)^{j-1} q^{(j-1)(2m-j)/2} \pmod{\Phi_m(q)}. \end{aligned}$$

Combined, the claim follows. □

5 Outlook and open problems

A major motivation for this paper is the observation of R. Osburn and B. Sahu [28] that all Apéry-like numbers appear to satisfy supercongruences. However, despite recent progress [29], it remains open to show that, for instance, the *Almkvist–Zudilin numbers* [4, sequence (4.12) (δ)], [11], [10]

$$Z(n)=\sum_{k=0}^{\lfloor n/3\rfloor}(-3)^{n-3k}\frac{(n+k)!}{(n-3k)!k!^4}, \tag{30}$$

satisfy the supercongruence

$$Z(p^r m)\equiv Z(p^{r-1}m) \pmod{p^{3r}} \tag{31}$$

for all primes $p\geq 3$. While the case $r>1$ remains open, T. Amdeberhan and R. Tauraso [5] recently proved the case $r=1$ of these congruences. It is not obvious how to introduce a q-analog of the Almkvist–Zudilin numbers $Z(n)$ in such a way that the congruence $Z(pm)\equiv Z(m)$ modulo p^3 has a polynomial analog of the kind proved in Corollary 1.1 for the q-Apéry numbers. On the other hand, finding a suitable q-analog of the numbers $Z(n)$ might provide some insight towards approaching the conjectured congruences (31) (and, even more optimistically, might lead to a better understanding of why such supercongruences hold for all of these sequences). In addition, as explained after Corollary 1.1, additional techniques exist for approaching polynomial supercongruences, including the *creative microscoping* approach recently discovered by V. Guo and W. Zudilin [22], who apply it with remarkable success to supercongruences in the context of Ramanujan-type formulae for $1/\pi$. It would be interesting to determine whether this approach, which involves (creatively!) introducing an additional parameter, can be applied to the congruences studied herein.

Returning to the specific case of the Almkvist–Zudilin numbers $Z(n)$, it was observed in [34] that they are the diagonal coefficients of the particularly simple rational function

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4}. \tag{32}$$

Recall that, similarly, the Apéry numbers are the diagonal coefficients of the rational function (21), and that we were able to use that fact to introduce natural q-analogs. However, in the case of the Almkvist–Zudilin numbers, our attempts to define a q-analog satisfying supercongruences like (3) have not been successful. A crucial difference is that MacMahon's Master Theorem does not apply as it did in the case of the Apéry number rational function.

It was shown by F. Beukers [9] that the Apéry numbers $A(n)$ occur as the coefficients when expanding the modular form

$$\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}=\sum_{n\geq 0}A(n)\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12n}. \tag{33}$$

Note that the expansion (33) is in terms of a modular function. As usual, $\eta(\tau)$ denotes the Dedekind eta function $\eta(\tau)=e^{\pi i\tau/12}\prod_{n\geq 1}(1-e^{2\pi i n\tau})$. Can this connection to modular forms be suitably extended to a q-analog of the Apéry numbers?

In another direction, Beukers [9] related the Apéry numbers to the pth Fourier coefficient $a(p)$ of the modular form $\eta^4(2\tau)\eta^4(4\tau)$, where $\eta(\tau)$ is the usual Dedekind eta function. He conjectured the congruence

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \tag{34}$$

and proved that (34) holds modulo p. The *modular supercongruence* (34) was later proved by S. Ahlgren and K. Ono [3], who reduce it to the identity

$$\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \{1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k\} = 0, \tag{35}$$

involving the harmonic sums $H_k = 1/1+1/2+\cdots+1/k$, which may be confirmed by the WZ method [2]. A classical proof of (35), by means of a partial fraction decomposition, has been given by W. Chu [13], who proves that

$$\frac{x(1-x)_n^2}{(x)_{n+1}^2} = \frac{1}{x} + \sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left\{ \frac{-k}{(x+k)^2} + \frac{1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k}{x+k} \right\},$$

which reduces to (35) in the limit. A q-analog of Chu's identity is obtained by D. Zheng [35]. In the limit, Zheng's result specializes to (20), which is a q-variation of (35) (taking a different limit, Zheng [35, Corollary 3] also gives a literal q-analog of (35)). It would be of considerable interest to find an appropriate setting for q-extending the supercongruence (34). Some pointers might be found in the recent work [22] of V. Guo and W. Zudilin on q-analogs of Ramanujan-type formulae for $1/\pi$.

A Proof of Theorem 2.2

The proof of Theorem 2.2 presented in this appendix is a straightforward extension of the corresponding proof given by the author in [33], from where most parts are taken literally.

It is well-known that the q-binomial coefficients have the following square-free factorization into cyclotomic polynomials. A less explicit statement appears in [14, Lemma 2], where the conclusion is emphasized that $\binom{n}{k}_q$ is a square-free polynomial which, for $k = 1, 2, \ldots, n-1$, is divisible by $\Phi_n(q)$.

Lemma A.1. *For nonnegative integers n, k,*

$$\binom{n}{k}_q = \prod_d \Phi_d(q), \tag{36}$$

where the product is taken over those $d \in \{2, 3, \ldots, n\}$ for which $\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor = 1$.

Proof. By definition, the q-number $[n]_q$ has the factorization

$$[n]_q = \prod_{d|n, d>1} \Phi_d(q).$$

Therefore,

$$[n]_q! = \prod_{m=1}^{n} \prod_{d|m, d>1} \Phi_d(q) = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor},$$

as well as

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}.$$

Clearly, $\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor \in \{0, 1\}$, so that we obtain (36). □

In preparation for the proof of Theorem 2.2, we show the following special case, to which the general case is then reduced.

Lemma A.2. *For integers $n \geq 0$,*

$$\binom{2n}{n}_q \equiv [2]_{q^{n^2}} - \frac{n^2-1}{12}(q^n - 1)^2 \pmod{\Phi_n(q)^3}. \tag{37}$$

Proof. Using that $[an]_q = [a]_{q^n}[n]_q$ and $[n+k]_q = [n]_q + q^n[k]_q$, we compute

$$\binom{2n}{n}_q = \frac{[2n]_q}{[n]_q} \prod_{k=1}^{n-1} \frac{[n+k]_q}{[k]_q} = [2]_{q^n} \prod_{k=1}^{n-1} \left(\frac{[n]_q}{[k]_q} + q^n \right),$$

which modulo $\Phi_n(q)^3$ reduces to (note that the terms $[k]_q$, with $k < n$, are relatively prime to $\Phi_n(q)$)

$$[2]_{q^n} \left(q^{(n-1)n} + q^{(n-2)n} \sum_{0<i<n} \frac{[n]_q}{[i]_q} + q^{(n-3)n} \sum_{0<i<j<n} \frac{[n]_q[n]_q}{[i]_q[j]_q} \right). \tag{38}$$

Generalizing an earlier result of Andrews [6], Shi and Pan prove in [32] and [30] that, for all positive integers n,

$$\sum_{0<i<n} \frac{1}{[i]_q} \equiv -\frac{n-1}{2}(q-1) + \frac{n^2-1}{24}(q-1)^2[n]_q \pmod{\Phi_n(q)^2}. \tag{39}$$

Note that this is a q-analog of Wolstenholme's well-known harmonic series congruence. Further, we have

$$\sum_{0<i<n} \frac{1}{[i]_q^2} \equiv -\frac{(n-1)(n-5)}{12}(q-1)^2 \pmod{\Phi_n(q)}, \tag{40}$$

which is shown, for prime n, in [32] and the proof extends readily to the case of composite n. Combining the two congruences (39) and (40), we conclude that

$$\sum_{0<i<j<n} \frac{1}{[i]_q[j]_q} \equiv \frac{(n-1)(n-2)}{6}(q-1)^2 \pmod{\Phi_n(q)}. \tag{41}$$

Applying the congruences (39) and (41) to (38), we obtain

$$\begin{aligned}\binom{2n}{n}_q \quad &\equiv \quad (1+q^n)\left(q^{(n-1)n} + q^{(n-2)n}\left(-\frac{n-1}{2}(q^n-1) + \frac{n^2-1}{24}(q^n-1)^2\right)\right. \\ &\left. \qquad + q^{(n-3)n}\frac{(n-1)(n-2)}{6}(q^n-1)^2\right) \pmod{\Phi_n(q)^3}.\end{aligned}$$

In order to reduce the terms q^{mn} modulo the appropriate power of $\Phi_n(q)$, we use the binomial expansion

$$q^{mn} = ((q^n-1)+1)^m = \sum_{k=0}^{m}\binom{m}{k}(q^n-1)^k,$$

This results in the simplified congruence

$$\binom{2n}{n}_q \equiv 2 + n(q^n-1) + \frac{(n-1)(5n-1)}{12}(q^n-1)^2 \pmod{\Phi_n(q)^3}. \tag{42}$$

It only remains to note that

$$[2]_{q^{n^2}} = 1 + q^{n^2} \equiv 2 + n(q^n-1) + \frac{n(n-1)}{2}(q^n-1)^2 \pmod{\Phi_n(q)^3},$$

in order to see that (42) is equivalent to the claimed congruence (37). □

We are now in a comfortable position to prove Theorem 2.2, which is restated below for the convenience of the reader.

Theorem A.3. *For positive integers n and nonnegative integers a, b,*

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}. \tag{43}$$

Proof. Observe that the two sides of (43) are trivially equal when $a = 0$ or $a = 1$. In the following, we therefore assume that $a \geq 2$. Then, as shown in [14], it follows from a q-analog of the Chu-Vandermonde convolution formula that

$$\binom{an}{bn}_q = \sum_{c_1+\ldots+c_a=bn} q^{n\sum_{1\leq i\leq a}(i-1)c_i - \sum_{1\leq i<j\leq a} c_ic_j}\binom{n}{c_1}_q\binom{n}{c_2}_q\cdots\binom{n}{c_a}_q. \tag{44}$$

By (36), the binomial coefficient $\binom{n}{k}_q$ is divisible by $\Phi_n(q)$ except in the boundary cases $k = 0$ and $k = n$. Therefore, the summands on the right-hand side of (44) which are not divisible by $\Phi_n(q)^2$ correspond to the c_i taking only the values 0 and n. Since each such summand is determined by the indices $1 \le j_1 < j_2 < \ldots < j_b \le a$ for which $c_i = n$, the total contribution of these terms is

$$\sum_{1\le j_1<\ldots<j_b\le a} q^{n^2 \sum_{k=1}^{b}(j_k-1)-n^2\binom{b}{2}} = \sum_{0\le i_1\le\ldots\le i_b\le a-b} q^{n^2\sum_{k=1}^{b} i_k} = \binom{a}{b}_{q^{n^2}}.$$

Note that we have arrived at (43) modulo $\Phi_n(q)^2$, that is

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} \pmod{\Phi_n(q)^2},$$

which is the main result of [14].

In order to prove the congruence (43), we now consider those summands in (44) which are divisible by $\Phi_n(q)^2$ but not divisible by $\Phi_n(q)^3$. These correspond to all but two of the c_i taking values 0 or n. More precisely, such a summand is determined by indices $1 \le j_1 < j_2 < \ldots < j_b < j_{b+1} \le a$, two subindices $1 \le k < \ell \le b+1$, and $1 \le d \le n-1$ such that

$$c_i = \begin{cases} d, & \text{for } i = j_k, \\ n-d, & \text{for } i = j_\ell, \\ n, & \text{for } i \in \{j_1,\ldots,j_{b+1}\}\backslash\{j_k,j_\ell\}, \\ 0, & \text{for } i \notin \{j_1,\ldots,j_{b+1}\}. \end{cases}$$

For each fixed choice of the j_i and k,ℓ, the contribution of the corresponding summands is

$$\sum_{d=1}^{n-1} q^{n\sum_{1\le i\le a}(i-1)c_i - \sum_{1\le i<j\le a} c_i c_j} \binom{n}{d}_q \binom{n}{n-d}_q. \tag{45}$$

Note that $q^n \equiv 1$ modulo $\Phi_n(q)$, and

$$\sum_{1\le i<j\le a} c_i c_j \equiv d(n-d) \equiv -d^2 \pmod{n}.$$

Since each binomial in (45) is divisible by $\Phi_n(q)$, we conclude that (45) reduces modulo $\Phi_n(q)^3$ to

$$\sum_{d=1}^{n-1} q^{d^2}\binom{n}{d}_q^2 = \binom{2n}{n}_q - [2]_{q^{n^2}},$$

where the equality follows from the special case $a = 2$ and $b = 1$ of (44). Since this does not depend on the value of d, and because there are

$$\binom{a}{b+1}\binom{b+1}{2} = \frac{(a-b)b}{2}\binom{a}{b}$$

choices for the j_i and k, ℓ, we find that

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} + \frac{(a-b)b}{2}\binom{a}{b}\left[\binom{2n}{n}_q - [2]_{q^{n^2}}\right] \pmod{\Phi_n(q)^3}.$$

The general result therefore follows from the special case $a = 2$, $b = 1$, which is proved in Lemma A.2. □

Acknowledgements. I am extremely thankful to Ofir Gorodetsky for motivating me to finally complete and write up the present paper, and for sharing a preprint of his work [20] in which he obtains q-congruences, from which he is able to derive the stronger congruences $A(p^r m) \equiv A(p^{r-1}m)$ modulo p^{3r} for the Apéry numbers due to Beukers and Coster. I am also grateful to Ofir Gorodetsky and Wadim Zudilin for helpful comments on an early draft of this paper.

References

[1] B. Adamczewski, J. P. Bell, E. Delaygue, and F. Jouhet. Congruences modulo cyclotomic polynomials and algebraic independence for q-series. *Sém. Lothar. Combin.*, 78B:Art. 54, 12, 2017.

[2] S. Ahlgren, S. B. Ekhad, K. Ono, and D. Zeilberger. A binomial coefficient identity associated to a conjecture of Beukers. *Electronic Journal of Combinatorics*, 5:#R10, 1 p., 1998.

[3] S. Ahlgren and K. Ono. A Gaussian hypergeometric series evaluation and Apéry number congruences. *Journal für die reine und angewandte Mathematik*, 2000(518):187–212, Jan. 2000.

[4] G. Almkvist, D. van Straten, and W. Zudilin. Generalizations of Clausen's formula and algebraic transformations of Calabi–Yau differential equations. *Proceedings of the Edinburgh Mathematical Society*, 54(2):273–295, 2011.

[5] T. Amdeberhan and R. Tauraso. Supercongruences for the Almkvist–Zudilin numbers. *Acta Arithmetica*, 173:255–268, 2016.

[6] G. E. Andrews. q-analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher. *Discrete Mathematics*, 204(1):15–25, 1999.

[7] R. Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

[8] F. Beukers. Some congruences for the Apéry numbers. *Journal of Number Theory*, 21(2):141–155, Oct. 1985.

[9] F. Beukers. Another congruence for the Apéry numbers. *Journal of Number Theory*, 25(2):201–210, Feb. 1987.

[10] H. H. Chan, S. Cooper, and F. Sica. Congruences satisfied by Apéry-like numbers. *International Journal of Number Theory*, 6(1):89, 2010.

[11] H. H. Chan and W. Zudilin. New representations for Apéry-like sequences. *Mathematika*, 56:107–117, 2010.

[12] S. Chowla, J. Cowles, and M. Cowles. Congruence properties of Apéry numbers. *Journal of Number Theory*, 12(2):188–190, May 1980.

[13] W. Chu. A binomial coefficient identity associated with Beukers' conjecture on Apéry numbers. *The Electronic Journal of Combinatorics*, 11(1):N15, Nov. 2004.

[14] W. E. Clark. q-analogue of a binomial coefficient congruence. *International Journal of Mathematics and Mathematical Sciences*, 18(1):197–200, 1995.

[15] M. J. Coster. *Supercongruences*. PhD thesis, Universiteit Leiden, 1988.

[16] J. Désarménien. Un analogue des congruences de Kummer pour les q-nombres d'Euler. *European Journal of Combinatorics*, 3(1):19–28, Mar. 1982.

[17] E. Deutsch and B. E. Sagan. Congruences for Catalan and Motzkin numbers and related sequences. *Journal of Number Theory*, 117(1):191–215, Mar. 2006.

[18] S. Formichella and A. Straub. Gaussian binomial coefficients with negative arguments. *Preprint*, Feb. 2018. arXiv:1802.02684.

[19] I. M. Gessel. Some congruences for Apéry numbers. *Journal of Number Theory*, 14(3):362–368, June 1982.

[20] O. Gorodetsky. q-congruences, with applications to supercongruences and the cyclic sieving phenomenon. *Preprint*, Mar. 2018.

[21] A. Granville. Arithmetic properties of binomial coefficients I: Binomial coefficients modulo prime powers. *CMS Conf. Proc.*, 20:253–275, 1997.

[22] V. J. W. Guo and W. Zudilin. A q-microscope for supercongruences. *Preprint*, Mar. 2018. arXiv:1803.01830.

[23] V. Kac and P. Cheung. *Quantum Calculus*. Springer-Verlag Universitytext, New York, 2002.

[24] C. Krattenthaler, T. Rivoal, and W. Zudilin. Séries hypergéométriques basiques, q-analogues des valeurs de la fonction zêta et séries d'Eisenstein. *J. Inst. Math. Jussieu*, 5(1):53–79, 2006.

[25] P. A. MacMahon. *Combinatory Analysis*. Cambridge Univ. Press, 1915.

[26] Y. Mimura. Congruence properties of Apéry numbers. *Journal of Number Theory*, 16(1):138–146, Feb. 1983.

[27] G. Olive. Generalized powers. *American Mathematical Monthly*, 72(6):619–627, June 1965.

[28] R. Osburn and B. Sahu. Congruences via modular forms. *Proceedings of the American Mathematical Society*, 139(7):2375–2381, 2011.

[29] R. Osburn, B. Sahu, and A. Straub. Supercongruences for sporadic sequences. *Proceedings of the Edinburgh Mathematical Society*, 59(2):503–518, 2016.

[30] H. Pan. A generalization of Wolstenholme's harmonic series congruence. *Rocky Mountain Journal of Mathematics*, 38(4):1263–1269, Aug. 2008.

[31] A. v. d. Poorten. A proof that Euler missed ... Apéry's proof of the irrationality of $\zeta(3)$. *Math. Intelligencer*, 1(4):195–203, 1979.

[32] L.-L. Shi and H. Pan. A q-analogue of Wolstenholme's harmonic series congruence. *Amer. Math. Monthly*, 114(6):529–531, 2007.

[33] A. Straub. A q-analog of Ljunggren's binomial congruence. *DMTCS Proceedings: 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, pages 897–902, Jun 2011. arXiv:1103.3258.

[34] A. Straub. Multivariate Apéry numbers and supercongruences of rational functions. *Algebra & Number Theory*, 8(8):1985–2008, 2014.

[35] D.-Y. Zheng. An algebraic identity on q-Apéry numbers. *Discrete Math.*, 311(2324):2708–2710, Dec. 2011.