CLOSED-FORM EVALUATION OF INTEGRALS APPEARING IN POSITRONIUM DECAY

TEWODROS AMDEBERHAN, VICTOR H. MOLL, AND ARMIN STRAUB

Abstract. A theoretical prediction for the total width of the positronium decay in QED has been given by B. Kniehl et al. in the form of an expansion in Sommerfeld's fine-structure constant. The coefficients of this expansion are given in the form of two-dimensional definite integrals, with an integrand involving the polylogarithm function. We provide here an analytic expression for the one-loop contribution to this problem.

1. Introduction

The single-scale problems in multi-loop analytic calculations from quantum field theories yield interesting classes of integrals. Some examples have appeared in the recent work by B. Kniehl et al [1] and [2] dealing with the lifetime of one of the two ground states of the *positronium*. This is the electromagnetic bound state of the electron e^- and the positron e^+. The main result of [2] is a theoretical prediction for the total width of positronium decay in QED given by

$$\Gamma(\text{theory}) = \Gamma_0 \left[1 + \frac{A\alpha}{\pi} + \frac{1}{3}\alpha^2 \ln \alpha + B \left(\frac{\alpha}{\pi} \right)^2 - \frac{3\alpha^3 \ln^2 \alpha}{2\pi} + \frac{C\alpha^3 \ln \alpha}{\pi} \right], \tag{1.1}$$

where α is Sommerfeld's fine-structure constant. The leading order term $\Gamma_0 = 2(\pi^2 - 9)m\alpha^6/9\pi$, as well as the $O\left(\alpha^2 \ln \alpha\right)$ and $O\left(\alpha^3 \ln^2 \alpha\right)$ terms are in the literature (with A, B, C in numerical form only). The remarkable contribution of [2] is to provide the first analytic expression for the coefficients A and C in (1.1). An analogous expression for B still remains to be completed. The formulas for A and C consist of a formidable collection of terms involving special values of $\ln x$, the Riemann zeta function $\zeta(x)$, the polylogarithm $\text{Li}_n(x)$ and the function

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \ln^{n-1} t \, \ln^p(1-tx)\, dt. \tag{1.2}$$

The explicit formulas can be found in [2].

The one-loop contribution to the width is given as

$$\Gamma_1 = \frac{m\alpha^7}{36\pi^2} \int_0^1 \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \delta(2 - x_1 - x_2 - x_3) \times \left[F(x_1, x_3) + \cdots \right], \tag{1.3}$$

where x_i, with $0 \le x_i \le 1$, is the normalized energy of the i-th photon and "..." represents F applied to each of the other five permutations of the variables. The

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evaluation of the integral (1.3) presents considerable analytic difficulties. After reparametrization, some terms in the function F involve integrals of the form

$$\begin{aligned} I_1(x_1,x_2) &= \int_0^1 \frac{\log[x_1+(1-x_1)y^2]}{(1-x_1)x_2-x_1(1-x_2)y^2}\,dy \\ I_2(x_1,x_2) &= \int_0^1 \frac{\log[x_1+(1-x_1)y^2]}{x_1x_2-(1-x_1)(1-x_2)y^2}\,dy. \end{aligned} \tag{1.4}$$

The goal of this note is to present an analytic evaluation of the integrals (1.4). This evaluation includes elementary functions as well as the *dilogarithm function*

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\log(1-t)}{t} dt. \tag{1.5}$$

Remark 1.1. D. Zagier states in [4] that ‘the dilogarithm is one of the simplest non-elementary functions. It is also one of the strangest. ... Almost all of its appearances in mathematics, and almost all formulas relating to it, have something of the fantastical in them, as if this function alone among all others possessed a sense of humor.’

The following basic relations are due to Euler:

$$\begin{aligned} \mathrm{Li}_2(z)+\mathrm{Li}_2(1-z) &= \frac{\pi^2}{6} - \log z\,\log(1-z), \\ \mathrm{Li}_2(z)+\mathrm{Li}_2(-z) &= \frac{1}{2}\mathrm{Li}_2(z^2), \\ \mathrm{Li}_2(z)+\mathrm{Li}_2(1/z) &= \frac{\pi^2}{3} - \frac{1}{2}\log^2(z) - i\pi\log z. \end{aligned}$$

Information about dilogarithms can be found in [3].

Notation. For $a \in \mathbb{R}$, we let $a^* := \dfrac{1-a}{1+a}$. Note that $(a^*)^* = a$, and $0 < a < 1$ if and only if $0 < a^* < 1$. For $a \in \mathbb{C}$, the condition $|a^*| \le 1$ is equivalent to $\mathrm{Re}\, a > 0$. The functions

$$\ell(a,b) = \mathrm{Li}_2\left(\frac{1-a}{1-b}\right) \tag{1.6}$$

and

$$\ell_s(a,b) = \ell(a,b) - \ell(-a,b) - \ell(a,-b) + \ell(-a,-b) \tag{1.7}$$

are used to give an analytic expression for the integrals I_1 and I_2.

Theorem 1.1. The positronium integrals are given by

$$\begin{aligned} I_1\left(\frac{1}{1-t_1^2}, \frac{1}{1-t_2^2}\right) &= -\frac{(1-t_1^2)(1-t_2^2)}{2t_1t_2}\left(\log t_1^*\,\log\left((t_2/t_1^2)^*\right) - \ell_s(t_1, t_1^2/t_2)\right), \\ I_2\left(\frac{1}{1-t_1^2}, \frac{1}{1-t_2^2}\right) &= \frac{(1-t_1^2)(1-t_2^2)}{2t_1t_2}\left(\log t_1^*\,\log t_2^* - \ell_s(t_1, 1/t_2)\right). \end{aligned}$$

Remark 1.2. Kummer’s formula for the dilogarithm [3] is

$$\begin{aligned} \mathrm{Li}_2\left(\frac{x(1-y)^2}{y(1-x)^2}\right) &= \mathrm{Li}_2\left(\frac{x(1-y)}{x-1}\right) + \mathrm{Li}_2\left(\frac{1-y}{y(x-1)}\right) \\ &+ \mathrm{Li}_2\left(\frac{x(1-y)}{y(1-x)}\right) + \mathrm{Li}_2\left(\frac{1-y}{1-x}\right) + \frac{1}{2}\log^2 y. \end{aligned}$$

A change of variable gives the identity

$$\ell(a,b)+\ell(-a,b)+\ell(a,-b)+\ell(-a,-b)=\ell(a^2,b^2)-\frac{1}{2}\log^2(-b^*) \tag{1.8}$$

and shows that $\ell_s(a,b)$ may be expressed as a sum of three dilogarithms plus elementary functions.

2. Some logarithmic integrals

The hypergeometric function

$$ {}_pF_q\left(\begin{matrix} a_1,\, a_2, \cdots, a_p \\ b_1,\, b_2, \cdots, b_q \end{matrix} ; z \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{2.1}$$

is now employed to establish the results in this section.

Lemma 2.1. For $a \neq b$

$$\int (1-ax)^{\lambda-1}(1-bx)^{\mu-1}dx = \frac{1}{\lambda}\frac{(1-ax)^{\lambda}(1-bx)^{\mu}}{b-a}\,{}_2F_1\left(\begin{matrix} 1, \lambda+\mu \\ \lambda+1 \end{matrix} ; \frac{1-ax}{1-a/b} \right).$$

Proof. This is verified by differentiation both sides with respect to x. □

Proposition 2.2. For $a \neq b$

$$\int_0^1 \frac{\log(1-ax)}{1-bx}\,dx = \frac{1}{b}\left[\mathrm{Li}_2\left(\frac{1}{1-a/b}\right) - \mathrm{Li}_2\left(\frac{1-a}{1-a/b}\right) - \log(1-a)\log\left(\frac{1-b}{1-b/a}\right)\right],$$

$$\int_0^1 \frac{\log(1-a^2x^2)}{1-b^2x^2}\,dx = \frac{1}{2b}\left[\ell_s(a,a/b) + \log a^* \log((b/a)^*) - \log b^* \log(1-a^2)\right].$$

Proof. Lemma 2.1 yields

$$\int \frac{(1-ax)^{\lambda-1}}{1-bx}dx = \frac{1}{\lambda}\frac{(1-ax)^{\lambda}}{b-a}\,{}_2F_1\left(\begin{matrix} 1, \lambda \\ \lambda+1 \end{matrix} ; \frac{1-ax}{1-a/b} \right).$$

Observe that

$$\frac{d}{d\lambda}\,{}_2F_1\left(\begin{matrix} 1, \lambda \\ \lambda+1 \end{matrix} ; z \right) = \frac{z}{(1+\lambda)^2}\,{}_3F_2\left(\begin{matrix} 2, \lambda+1, \lambda+1 \\ \lambda+2,\, \lambda+2 \end{matrix} ; z \right). \tag{2.2}$$

Differentiating with respect to λ leads to

$$\int \log(1-ax)\frac{(1-ax)^{\lambda-1}}{1-bx}\,dx = \frac{1}{\lambda}\frac{(1-ax)^{\lambda}}{b-a}\left[\left(\log(1-ax) - \frac{1}{\lambda}\right){}_2F_1\left(\begin{matrix} 1, \lambda \\ \lambda+1 \end{matrix} ; \frac{1-ax}{1-a/b} \right) + \frac{1-ax}{(1-a/b)(1+\lambda)^2}\,{}_3F_2\left(\begin{matrix} 2, \lambda+1, \lambda+1 \\ \lambda+2,\, \lambda+2 \end{matrix} ; \frac{1-ax}{1-a/b} \right)\right].$$

Now set $\lambda = 1$ and use $\mathrm{Li}_1(z) = -\log(1-z)$, as well as

$$ {}_3F_2\left(\begin{matrix} 2,\, 2,\, 2 \\ 3,\, 3 \end{matrix} ; z \right) = -\frac{4}{z^2}\left[\log(1-z) + \mathrm{Li}_2(z)\right] \tag{2.3}$$

to establish the first claim. The factorization $(1-a^2x^2) = (1-ax)(1+ax)$ and the partial fraction decomposition

$$\frac{2}{1-b^2x^2} = \frac{1}{1-bx} + \frac{1}{1+bx} \tag{2.4}$$

give the second evaluation. □

3. A trigonometric integral

The results in Section 2 provide the value of an interesting trigonometric integral in terms of Legendre's χ_2 function

$$\chi_2(a) := \frac{1}{2}\left(\mathrm{Li}_2(a) - \mathrm{Li}_2(-a)\right). \tag{3.1}$$

Proposition 3.1. For $a \in \mathbb{R}$

$$\int_b^\infty \frac{\tan^{-1}(ax)}{1+x^2}\,dx = \chi_2(a) + \frac{1}{2}\log a \log a^* + \frac{1}{4}\ell_s(a, i/b). \tag{3.2}$$

Proof. Observe that

$$\begin{aligned}\frac{d}{da}\int_b^\infty \frac{\tan^{-1}(ax)\,dx}{1+x^2} &= \int_b^\infty \frac{x\,dx}{(1+a^2x^2)(1+x^2)} \\ &= \frac{1}{2(1-a^2)}\left(\log(1+a^2b^2) - 2\log a - \log(1+b^2)\right).\end{aligned}$$

The original integral is recovered via

$$\int_0^a \frac{ds}{1-s^2} = \frac{1}{2}\log\left(\frac{1+a}{1-a}\right) = -\frac{1}{2}\log a^*, \tag{3.3}$$

as well as

$$\int_0^a \frac{2\log s\,ds}{1-s^2} = \mathrm{Li}_2(1-a) - \mathrm{Li}_2(1) + \mathrm{Li}_2(-a) + \log a\,\log(1+a). \tag{3.4}$$

The last term to evaluate

$$\int_0^a \frac{\log(1+s^2b^2)}{1-s^2}\,ds = a\int_0^1 \frac{\log(1+a^2b^2x^2)}{1-a^2x^2}\,dx, \tag{3.5}$$

is given by Proposition 2.2 as

$$\frac{1}{2}\left[\ell_s(iab, ib) + \log((iab)^*)\log(-(ib)^*) - \log(a^*)\log(1+a^2b^2)\right]. \tag{3.6}$$

The result now follows from Euler's transformations for the dilogarithm given after Remark 1.1. ∎

Letting $b \to 0$ produces the integral over the half-line.

Corollary 3.2. The evaluation

$$\int_0^\infty \frac{\tan^{-1}(ax)\,dx}{1+x^2} = \chi_2(a) + \frac{1}{2}\log a \log(a^*). \tag{3.7}$$

holds.

4. Application to the positronium decay integrals

For the convenience of the reader we reproduce Theorem 1.1:

Theorem 4.1. The positronium integrals are given by

$$\begin{aligned}I_1\left(\frac{1}{1-t_1^2}, \frac{1}{1-t_2^2}\right) &= -\frac{(1-t_1^2)(1-t_2^2)}{2t_1t_2}\left(\log t_1^*\,\log\left((t_2/t_1^2)^*\right) - \ell_s(t_1, t_1^2/t_2)\right), \\ I_2\left(\frac{1}{1-t_1^2}, \frac{1}{1-t_2^2}\right) &= \frac{(1-t_1^2)(1-t_2^2)}{2t_1t_2}\left(\log t_1^*\,\log t_2^* - \ell_s(t_1, 1/t_2)\right).\end{aligned}$$

Proof. The integral $I_1(x_1, x_2)$ is written as

$$\frac{-t_1^2}{(1-t_1^2)(1-t_2^2)} I_1\left(\frac{1}{1-t_1^2}, \frac{1}{1-t_2^2}\right) = \int_0^1 \log\left(\frac{1-t_1^2 y^2}{1-t_1^2}\right) \frac{dy}{1-(t_2/t_1)^2 y^2}. \tag{4.1}$$

Proposition 2.2 yields

$$\begin{aligned}\int_0^1 \log\left(\frac{1-t_1^2 y^2}{1-t_1^2}\right) \frac{dy}{1-(t_2/t_1)^2 y^2} &= \int_0^1 \frac{\log(1-t_1^2 y^2)}{1-(t_2/t_1)^2 y^2}\, dy - \int_0^1 \frac{\log(1-t_1^2)}{1-(t_2/t_1)^2 y^2}\, dy \\ &= \frac{t_1}{2t_2}\left[\log t_1^* \log((t_2/t_1^2)^*) - \ell_s(t_1, t_1^2/t_2)\right].\end{aligned}$$

The second positronium integral is evaluated analogously. □

The following special case is recorded.

Corollary 4.2. Assume $0 < a < 1$. Then

$$\int_0^1 \frac{\log(a+(1-a)x^2)}{1-x^2}\, dx = -\arctan^2\left(\sqrt{\frac{1-a}{a}}\right). \tag{4.2}$$

Proof. Let $a = 1/(1-t^2)$. Then

$$\begin{aligned}\int_0^1 \frac{\log(a+(1-a)x^2)}{1-x^2}\, dx &= a(1-a) I_1(a,a) \\ &= \frac{1}{2}\left[\log t^* \log((1/t)^*) - \ell_s(t,t)\right].\end{aligned}$$

It follows from Remark 1.1 that

$$\ell_s(t,t) = \frac{\pi^2}{3} - \mathrm{Li}_2\left(\frac{1-t}{1+t}\right) - \mathrm{Li}_2\left(\frac{1+t}{1-t}\right) = \frac{1}{2}\log^2 t^* + i\pi \log t^*. \tag{4.3}$$

Thus

$$\int_0^1 \frac{\log(a+(1-a)x^2)}{1-x^2}\, dx = \left(\frac{1}{2}\log t^*\right)^2 \tag{4.4}$$

and this is (4.2). □

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Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: tamdeberhan@math.tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: astraub@math.tulane.edu