ON MULTIPLE AND INFINITE LOG-CONCAVITY

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Abstract. Following Boros–Moll, a sequence (a_n) is m-log-concave if $\mathcal{L}^j(a_n) \geqslant 0$ for all $j = 0, 1, \ldots, m$. Here, $\mathcal{L}$ is the operator defined by $\mathcal{L}(a_n) = a_n^2 - a_{n-1}a_{n+1}$. By a criterion of Craven–Csordas and McNamara–Sagan it is known that a sequence is ∞-log-concave if it satisfies the stronger inequality $a_k^2 \geqslant r a_{k-1}a_{k+1}$ for large enough r. On the other hand, a recent result of Brändén shows that ∞-log-concave sequences include sequences whose generating polynomial has only negative real roots. In this paper, we investigate sequences which are fixed by a power of the operator $\mathcal{L}$ and are therefore ∞-log-concave for a very different reason. Surprisingly, we find that sequences fixed by the non-linear operators $\mathcal{L}$ and $\mathcal{L}^2$ are, in fact, characterized by a linear 4-term recurrence. In a final conjectural part, we observe that positive sequences appear to become ∞-log-concave if convoluted with themselves a finite number of times.

1. Introduction

A sequence (a_n) is said to be log-concave if $a_n^2 \geqslant a_{n-1}a_{n+1}$ for all n. If all terms of the sequence are positive, then log-concavity implies unimodality. For a very nice introduction and many examples of both unimodal and log-concave sequences we refer to [17].

Following Boros and Moll [2], we denote with $\mathcal{L}$ the operator which sends a sequence (a_n) to the sequence $(a_n^2 - a_{n-1}a_{n+1})$. Then (a_n) is log-concave if and only if $\mathcal{L}(a_n) \geqslant 0$. Similarly, the sequence (a_n) is said to be m-log-concave if $\mathcal{L}^j(a_n) \geqslant 0$ for all $j = 0, 1, \ldots, m$. If (a_n) is m-log-concave for all $m > 0$, then it is said to be ∞-log-concave. Often, we will consider the generating function $f(x) = \sum_{n \geqslant 0} a_n x^n$. In that case, we write $\mathcal{L}[f](x) = \sum_{n \geqslant 0}(a_n^2 - a_{n-1}a_{n+1})x^n$ with the understanding that $a_{-1} = 0$.

Example 1.1. The sequence $\binom{n}{2}$ is fixed by the operator $\mathcal{L}$ because

$$\binom{n}{2}^2 - \binom{n-1}{2}\binom{n+1}{2} = \binom{n}{2}.$$

Since the sequence is nonnegative, it is therefore ∞-log-concave. In Section 3, we will characterize all sequences fixed by $\mathcal{L}$.

More generally, as a warm-up problem, it was conjectured in [2] that the binomial coefficients (either rows, that is $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$, or columns, that is $\binom{n}{n}, \binom{n+1}{n}, \binom{n+2}{n}, \ldots$, of Pascal's triangle) are ∞-log-concave. That the rows of Pascal's triangle are indeed ∞-log-concave was recently proven by Brändén [3] who established, much more generally, that the coefficients of polynomials, all of whose roots are negative

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and real, are always ∞-log-concave. This connection of the location of roots and log-concavity will be briefly reviewed in Section 2. We remark that, on the other hand, the case of columns of Pascal's triangle is still wide open; based on extensive computations by Kauers and Paule [12] it is only known that they are 5-log-concave.

As will also be reviewed in Section 2, a sequence is ∞-log-concave if it satisfies the stronger condition $a_n^2 \geqslant r a_{n-1} a_{n+1}$ for $r \geqslant (3+\sqrt{5})/2 \approx 2.618$. This criterion, due to Craven and Csordas [5] as well as McNamara and Sagan [14], generalizes to a powerful approach of showing that a specific sequence is ∞-log-concave. On the other hand, there are sequences, like the one in Example 1.1, that are fixed by the operator $\mathcal{L}$ (or one of its powers) and are ∞-log-concave for this reason. This leads us to investigate the sequences fixed by $\mathcal{L}$ in Section 3 as well as the sequences fixed by $\mathcal{L}^2$ in Section 4. In both cases, we find that, surprisingly, these sequences fixed by non-linear operators are characterized by *linear* 4-term recurrences with constant coefficients. This phenomenon does not appear to extend to the sequences fixed by $\mathcal{L}^m$ for $m > 2$. As an application, we ask if these criteria for ∞-log-concavity can be combined to yield an algorithm that decides, in finite time, whether or not a given finite sequence is ∞-log-concave.

In the final section, Section 5, we start with positive sequences and repeatedly convolute them with themselves. That is to say, we consider the coefficients of powers of a given polynomial. The central limit theorem suggests that, as the exponent increases, the shape of the resulting sequences approaches the shape of a normal distribution. One therefore expects the sequences to become more and more log-concave. Indeed, we observe that each sequence appears to become ∞-log-concave in a finite number of steps.

2. Review of multiple log-concavity

Newton's famous and classical theorem on real roots states that if the polynomial $p(x) = a_0 + a_1 x + \ldots + a_d x^d$ has only negative real roots, then its coefficients (a_n) are log-concave. We refer to [15] for historic information and related results.

Remark 2.1. In fact, somewhat stronger, Newton's theorem implies that the numbers $a_n / \binom{d}{n}$ form a log-concave sequence. This is indeed stronger since the binomial coefficients are log-concave and the Hadamard product of two log-concave sequences is again log-concave. If only log-concavity is desired, the assumptions of Newton's theorem can be weakened: it is shown in [1] that the condition that all roots are negative (and therefore real) can be replaced with the condition that all roots lie in the sector defined by $|\arg(-z)| \leqslant \pi/3$ (equivalently, all roots $z = a + bi$ satisfy both $a \leqslant 0$ and $b^2 \leqslant 3a^2$).

Brändén [3] recently established the following more general theorem which was previously, and independently, conjectured by Stanley, McNamara–Sagan [14] and Fisk [7].

Theorem 2.2. ([3]) *Let $p(x) = \sum_{n=0}^{d} a_n x^n$. If all roots of p are negative and real then so are the roots of*

$$\mathcal{L}[p](x) = \sum_{n=0}^{d} (a_n^2 - a_{n-1} a_{n+1}) x^n.$$

Here, it is understood that $a_{-1} = 0$ as well as $a_{d+1} = 0$.

Corollary 2.3. *If the polynomial $p(x) = a_0 + a_1x + \ldots + a_dx^d$ has only negative real roots, then its coefficients (a_n) are ∞-log-concave.*

Example 2.4. If $a_n = \binom{d}{n}$, then $p(x) = \sum_{n=0}^{d} a_nx^n = (x+1)^d$ clearly has only negative real roots. It follows that (rows of) the binomial coefficients are ∞-log-concave, confirming the motivating conjecture in [2].

Note that the converse of Corollary 2.3 is not true, as is illustrated by the polynomial

$$P_2(x) = \frac{3}{2}x^2 + \frac{15}{4}x + \frac{21}{8},$$

which has non-real roots but coefficients which are ∞-log-concave; this is easily proved using the notion of r-factor log-concavity which is reviewed next. We remark that $P_2(x)$ is one of the Boros–Moll polynomials, which occurred in the evaluation of a quartic integral [2], and have been investigated by many authors hence; we only refer to the recent article [4] as well as the references therein. Nevertheless, as illustrated by Theorem 2.8 below, a partial converse to Brändén's result is possible if ∞-log-concavity is replaced by an even stronger property.

An important tool for establishing infinite log-concavity of a specific sequence is the notion of r-factor log-concavity, which has been introduced and used by McNamara and Sagan [14]. Let $r \geqslant 1$. A sequence (a_n) is r-factor log-concave if

$$a_n^2 \geqslant r a_{n-1}a_{n+1}.$$

In fact, using slightly different terminology, this notion has already been considered by Craven and Csordas in [5]. Let $\mathcal{M}_r$ be the r-factor log-concave sequences. It is shown in [5, Theorem 4.1] that

$$\mathcal{L}(\mathcal{M}_r) \subset \mathcal{M}_{r+s} \tag{1}$$

if and only if $r \geqslant \frac{3+\sqrt{5+4s}}{2}$. Generalizations, for instance to the case of decreasing sequences, appear in [8]. Of particular importance for our purposes is the following consequence of the containment (1).

Lemma 2.5. *If a sequence (a_n) is r-factor log-concave for some $r \geqslant \frac{3+\sqrt{5}}{2} \approx 2.618$, then $\mathcal{L}(a_n)$ is r-factor log-concave as well. In particular, the sequence (a_n) is ∞-log-concave.*

As illustrated in the next example, this gives rise to a more general approach, used in [14], to show that a specific sequence is ∞-log-concave.

Example 2.6. The polynomial $p(x) = 1 + 4x + 6x^2 + 4x^3$ has coefficients which are 2.25-factor log-concave. The polynomial $\mathcal{L}[p](x) = 1 + 10x + 20x^2 + 16x^3$ has coefficients which are 2.5-factor log-concave, while the coefficients of $\mathcal{L}^2[p](x) = 1 + 80x + 240x^2 + 256x^3$ are 2.8125-factor log-concave. Lemma 2.5 therefore shows that, in fact, the coefficients of $p(x)$ are ∞-log-concave.

The concept of r-factor log-concavity of the coefficients of a polynomial is intimately related to the location of zeros discussed earlier in this section. For instance, the following result on Hurwitz stability is due to Katkova and Vishnyakova. We refer to [11] for further details, including the cases of lower degree, and references to related and earlier results. Here, *strong* log-concavity, and likewise r-factor strong log-concavity, mean that the defining inequalities are strict (for instance, a sequence (a_n) is r-factor strongly log-concave if $a_n^2 > r a_{n-1}a_{n+1}$).

Theorem 2.7. ([11]) *Let $p(x)$ be a polynomial with positive coefficients and degree larger than 5. If the coefficients of $p(x)$ are r_0-factor strongly log-concave, where $r_0 \approx 1.466$ is the unique real root of $r^3 - r^2 - 1$, then all the roots of $p(x)$ have negative real parts.*

As a second example, we cite the following result of Kurtz [13], as stated in [6], which may be viewed as a partial converse to Newton's theorem on real roots.

Theorem 2.8. ([13]) *Let $p(x)$ be a polynomial with positive coefficients. If the coefficients of $p(x)$ are 4-factor log-concave, then all the roots of $p(x)$ are real (and hence negative).*

It is also shown in [13] that 4-factor log-concavity cannot be replaced $(4 - \varepsilon)$-factor log-concavity for any $\varepsilon > 0$.

3. Sequences fixed by $\mathcal{L}$

As observed in Example 1.1, the sequence $\binom{n}{2}$ is fixed by $\mathcal{L}$ and therefore ∞-log-concave. In this section, we characterize all sequences $(a_n)_{n\geqslant 0}$ that are fixed by the operator $\mathcal{L}$, that is $\mathcal{L}(a_n) = (a_n)$, or, equivalently,

$$a_n^2 - a_{n-1}a_{n+1} = a_n \tag{2}$$

for all indices $n \geqslant 0$ (with the understanding that $a_{-1} = 0$).

Remark 3.1. Let us note that the, apparently more general, characterization of sequences $(a_n)_{n\geqslant 0}$ such that $\mathcal{L}(a_n) = \lambda(a_n)$ for some number λ reduces to the case $\lambda = 1$. Indeed, if $\mathcal{L}(a_n) = \lambda(a_n)$ then $\mathcal{L}(b_n) = (b_n)$ with $b_n = a_n/\lambda$.

Suppose that $(a_n)_{n\geqslant 0}$ is a sequence fixed by $\mathcal{L}$. Note that if $a_{m-1} \neq 0$ and $a_m = 0$ for some $m > 0$, then $a_{m+1} = 0$ as well. In particular, the sequence $(a_{n+m+2})_{n\geqslant 0}$ is again fixed by $\mathcal{L}$. In characterizing all sequences that are fixed by $\mathcal{L}$, it is therefore no loss of generality to assume that the sequence (a_n) has no *internal zeros*, meaning that if $a_m = 0$ for some $m \geqslant 0$ then $a_n = 0$ for all $n \geqslant m$. If (a_n) has no internal zeros, then $a_0 = 1$ unless (a_n) is the zero sequence. Assuming that (a_n) is not the zero sequence, the value of a_1 then determines (a_n).

Example 3.2. With the value k given, let $(a_n)_{n\geqslant 0}$ be the unique sequence with $a_1 = k$ which is fixed by $\mathcal{L}$ and has no internal zeros. When $k \in \{0, 1, 2\}$ this sequence is finitely supported with values (1), $(1, 1)$ and $(1, 2, 2, 1)$, respectively. On the other hand, if $k \geqslant 3$ is an integer, then the corresponding sequence is infinite and all of its terms are positive integers. These claims are not *a priori* obvious but will be direct consequences of the characterization in Theorem 3.3. The first few cases, with $k = 3, 4, 5$, are:

$$1, 3, 6, 10, 15, 21, 28, 36, \ldots$$
$$1, 4, 12, 33, 88, 232, 609, 1596, \ldots$$
$$1, 5, 20, 76, 285, 1065, 3976, 14840, \ldots$$

In the first case, corresponding to $k = 3$, the sequence is given by $a_n = \binom{n+2}{2}$, the introductory Example 1.1, and in the case $k = 4$ we identify the sequence as $a_n = F_{2n+3} - 1$ with F_n denoting the Fibonacci numbers.

The alert reader may have noticed that the infinite sequences given in Example 3.2 each have a rational generating function. The next result and its corollary show that this is always the case.

Theorem 3.3. *Let k be an arbitrary number. Then the sequence $(a_n)_{n\geqslant 0}$ defined by*

$$\sum_{n\geqslant 0} a_n x^n = \frac{1}{1-kx+kx^2-x^3} = \frac{1}{(1-x)(1-(k-1)x+x^2)}$$

is fixed by $\mathcal{L}$. Note that $a_0 = 1$, $a_1 = k$.

Proof. Let S be the (inverse) shift operator defined by $Sa_n = a_{n-1}$, and consider the operator

$$L := 1 - kS + kS^2 - S^3 = (1-S)\left(1-(k-1)\,S+S^2\right). \tag{3}$$

We note that the rational generating function for (a_n) is equivalent to the fact that the recurrence $La_n = 0$ holds, for all $n \geqslant 1$, with initial conditions $a_0 = 1$ and $a_{-1} = a_{-2} = 0$.

The factorization of L, as on the right-hand side of (3), implies that the sequence $b_n = \left(1-(k-1)\,S+S^2\right) a_n = a_n - (k-1)\,a_{n-1} + a_{n-2}$ is constant. Since $b_0 = a_0 = 1$ it follows that $b_n = 1$ for all $n \geqslant 0$. In other words, (a_n) satisfies the nonhomogeneous recurrence

$$a_n - (k-1)\,a_{n-1} + a_{n-2} = 1 \tag{4}$$

for all $n \geqslant 0$.

To prove that (a_n) is fixed under $\mathcal{L}$, we need to show that

$$a_n\,(a_n - 1) = a_{n+1}a_{n-1} \tag{5}$$

for all $n \geqslant 0$. Equation (5) clearly holds for $n = 0$. For the purpose of induction, assume that (5) holds for some n. Then

$$\begin{aligned}
a_{n+1}\,(a_{n+1}-1) &\overset{(4)}{=} a_{n+1}\,((k-1)\,a_n - a_{n-1}) \\
&= (k-1)\,a_{n+1}a_n - a_{n+1}a_{n-1} \\
&\overset{(5)}{=} (k-1)\,a_{n+1}a_n - a_n\,(a_n-1) \\
&= a_n\,((k-1)\,a_{n+1} - (a_n-1)) \\
&\overset{(4)}{=} a_{n+2}a_n,
\end{aligned}$$

which shows that (5) also holds for $n+1$ in place of n. The result therefore follows by induction. □

Remark 3.4. Note that the sequence (a_n), defined in Theorem 3.3, is C-finite [18]. Moreover, since C-finite sequences form an algebra, the sequences (a_n^2) and $(a_{n-1}a_{n+1})$, as well as any linear combination of these, are again C-finite. In order to obtain an alternative and automatic proof of Theorem 3.3, we can therefore use the C-finite ansatz, recently advertised in [19], to show that

$$a_n^2 - a_{n-1}a_{n+1} = a_n,$$

thus proving that (a_n) is indeed fixed by $\mathcal{L}$.

Theorem 3.3 has the, possibly surprising, consequence that the solutions of the non-linear three-term recurrence (2) in fact satisfy a linear four-term recurrence.

Corollary 3.5. *Let $N \in \mathbb{N} \cup \{\infty\}$. Suppose that the sequence $(a_n)_{n=0}^N$ is fixed by $\mathcal{L}$ and that $a_n \neq 0$ for all $0 \leqslant n \leqslant N$. Then, with $k = a_1$,*

$$a_n - k(a_{n-1} - a_{n-2}) - a_{n-3} = 0 \tag{6}$$

for all $n = 1, 2, \ldots, N$, with initial conditions $a_0 = 1$ and $a_{-1} = a_{-2} = 0$.

Proof. Recall that if (a_n) is fixed by $\mathcal{L}$ and $a_0 \neq 0$, then $a_0 = 1$ and the value of a_1 determines the initial segment of nonzero terms of (a_n). On the other hand, for any value $k = a_1$, Theorem 3.3 provides a sequence that is fixed by $\mathcal{L}$. It follows that the two sequences have to agree for all initial nonzero terms. □

As illustrated by Example 3.2, the sequences fixed by $\mathcal{L}$ usually have infinite support. We now determine all sequences with finite support that are fixed by $\mathcal{L}$.

Proposition 3.6. *Let $s \geqslant 3$ and $1 \leqslant r < s$ be integers. If $s \neq 2r$, then*

$$p_{s,r}(x) = \frac{1 - x^s}{(1-x)(1 - 2\cos(2\pi r/s)x + x^2)} \tag{7}$$

is a degree $s - 3$ polynomial, and the coefficients of $p_{s,r}$ are fixed by $\mathcal{L}$. Moreover, every finitely supported sequence which is fixed by $\mathcal{L}$ and has no internal zeros arises in this way.

Proof. Assume that (a_n) is one of the sequences of Theorem 3.3 with the property that there is $N \geqslant 0$ such that $a_{N+1} = 0$. Choose the minimal such N. Then

$$(a_n)_{n \geqslant 0} = (1, a_1, \ldots, a_{N-1}, 1, 0, 0, 1, a_1, \ldots).$$

Conversely, consider a finite sequence $(a_n)_{n=0}^N$ of nonzero terms which is fixed by $\mathcal{L}$ (so that, in particular, $a_0 = a_N = 1$). Then $p(x) = a_0 + a_1 x + \ldots + a_N x^N$ necessarily satisfies

$$\frac{p(x)}{1 - x^{N+3}} = \frac{1}{(1-x)(1 - (a_1 - 1)x + x^2)}.$$

It follows that $1 - (a_1 - 1)x + x^2 = (x - \zeta_1)(x - \zeta_2)$ where ζ_1, ζ_2 are nontrivial s-th roots of unity, with $s = N + 3$. In fact, one clearly has $\zeta_2 = \zeta_1^{-1} = \overline{\zeta_1}$. Writing $\zeta_1 = \exp(2\pi i r/s)$ for some r, we thus find

$$a_1 - 1 = \zeta_1 + \overline{\zeta_1} = 2\cos\left(\frac{2\pi r}{s}\right).$$

The claims follow from here. □

We observe that the polynomials $p_{s,r}(x)$ are palindromic, that is, their coefficients form symmetric sequences. This property follows from $x^{s-3} p_{s,r}(1/x) = p_{s,r}(x)$ which is readily verified from (7). While the coefficients of these polynomials are not in general positive, we now show they are positive in the case $r = 1$. In this case, the coefficients therefore form an interesting family of sequences which are ∞-log-concave; see Example 3.8 which lists the first few sequences explicitly.

Theorem 3.7. *Let $s \geqslant 3$ be an integer. The polynomials*

$$p_s(x) = p_{s,1}(x) = \frac{1 - x^s}{(1-x)(1 - 2\cos(2\pi/s)x + x^2)} \tag{8}$$

have the following properties:

(a) The polynomial $p_s(x)$ is palindromic of degree $s - 3$.
(b) The coefficients of $p_s(x)$ are positive.
(c) The coefficients of $p_s(x)$ are fixed by $\mathcal{L}$.
(d) The coefficients of $p_s(x)$ are ∞-log-concave.

(e) The coefficients of $p_s(x)$ are r-factor log-concave if and only if

$$r \leqslant \begin{cases} \frac{1}{\cos(2\pi/s)}, & \text{if } s \text{ is even,} \\ \frac{1}{[1-2\cos(\pi/s)]^2}, & \text{if } s \text{ is odd.} \end{cases}$$

Proof. Part (a) is a special case of the palindromicity of the polynomials $p_{s,r}(x)$ which was observed above. Note that claim (c) was proved in Proposition 3.6. Together with the positivity claimed in part (b), this implies the ∞-log-concavity of part (d). It therefore only remains to show parts (b) and (e).

Let us prove (b). Since the polynomial $p_s(x) = a_0 + a_1 x + \ldots + a_{s-3}x^{s-3}$ is palindromic of degree $s-3$, it is sufficient to show that $a_j > 0$ for $j = 0, 1, \ldots, k$ where $k = \lfloor (s-3)/2 \rfloor$. We recall the classical generating function

$$\frac{1}{1 - 2zx + x^2} = \sum_{n=0}^{\infty} U_n(z) x^n$$

of the Chebyshev polynomials $U_n(z)$ of the second kind. Since $U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$, we have

$$\frac{1}{1 - 2\cos(\theta)x + x^2} = \frac{1}{\sin(\theta)} \sum_{n=0}^{\infty} \sin((n+1)\theta) x^n$$

and hence

$$p_s(x) = \frac{1 + x + \ldots + x^{s-1}}{\sin(2\pi/s)} \sum_{n=0}^{\infty} \sin(2\pi(n+1)/s) x^n. \tag{9}$$

Note that $\sin(2\pi/s) > 0$. The coefficient a_j therefore is a positive linear combination of $\sin(2\pi(n+1)/s)$ for $n = 0, 1, \ldots, j$. On the other hand, for $n = 0, 1, \ldots, k$,

$$\frac{2\pi(n+1)}{s} \leqslant \frac{2\pi(k+1)}{s} \leqslant \frac{\pi(s-1)}{s} < \pi$$

and hence $\sin(2\pi(n+1)/s) > 0$. Therefore, the coefficients of $p_s(x)$ are positive as claimed in (b).

To show part (e), recall that (a_n) is r-factor log-concave if and only if $a_n^2 - r a_{n+1} a_{n-1} \geqslant 0$ for all $n = 1, 2, \ldots, s-4$. Using the fact that $a_n^2 - a_{n+1}a_{n-1} = a_n$, this inequality is clearly equivalent to

$$r \leqslant \frac{a_n^2}{a_{n+1}a_{n-1}} = \frac{a_n}{a_n - 1}.$$

We have already shown that the coefficients a_n are positive and fixed by $\mathcal{L}$. Hence they are log-concave and, as a consequence, unimodal. Because the sequence (a_n) is symmetric and unimodal, its maximum is a_N, with $N = \lfloor (s-3)/2 \rfloor$, and it follows that (a_n) is r-factor log-concave if and only if $r \leqslant \frac{a_N}{a_N - 1}$.

The value a_N can be obtained from the expansion (9). Indeed, writing $\zeta_s = e^{2\pi i/s}$, we find that, for $n = 0, 1, \ldots, s-3$,

$$a_n - 1 = \frac{1}{\sin(2\pi/s)} \operatorname{Im} \sum_{j=1}^{n} \zeta_s^{j+1} = \frac{1}{\sin(2\pi/s)} \operatorname{Im} \left[\zeta_s^2 \frac{1 - \zeta_s^n}{1 - \zeta_s} \right].$$

A simple calculation shows that

$$\begin{aligned} \operatorname{Im}\left[\zeta_s^2\frac{1-\zeta_s^n}{1-\zeta_s}\right] &= \operatorname{Im}\left[\frac{-\zeta_s+\zeta_s^2+\zeta_s^{n+1}-\zeta_s^{n+2}}{|1-\zeta_s|^2}\right] \\ &= \frac{-\sin(\frac{2\pi}{s})+\sin(\frac{4\pi}{s})+\sin(\frac{2\pi(n+1)}{s})-\sin(\frac{2\pi(n+2)}{s})}{2-2\cos(2\pi/s)}. \end{aligned}$$

In the case when n is $N=\lfloor (s-3)/2\rfloor$, the sines can all be expressed with arguments in terms of s only, and one obtains

$$a_N-1=\frac{1}{\sin(2\pi/s)}\begin{cases}\frac{\sin(4\pi/s)}{2-2\cos(2\pi/s)}, & \text{if } s \text{ is even,}\\ \frac{2\sin(\pi/s)-\sin(2\pi/s)+\sin(4\pi/s)}{2-2\cos(2\pi/s)}, & \text{if } s \text{ is odd.}\end{cases}$$

Using $\frac{a_N}{a_N-1}=1+\frac{1}{a_N-1}$, it is now straightforward to verify the claim using standard trigonometric identities. Indeed, since $\cos(2t)=2\cos(t)^2-1$, we have, by definition of the Chebyshev polynomials U_n,

$$a_N-1=\frac{1}{4(1-t^2)U_1(t)}\begin{cases}U_3(t), & \text{if } s \text{ is even,}\\ 2U_0(t)-U_1(t)+U_3(t), & \text{if } s \text{ is odd,}\end{cases}$$

with $t=\cos(\pi/s)$, from which the claim is immediate. □

Example 3.8. The first few polynomials of Theorem 3.7 are $p_3(x)=1$, $p_4(x)=1+x$, as well as

$$\begin{aligned} p_5(x) &= 1+\tfrac{1+\sqrt{5}}{2}x+x^2,\\ p_6(x) &= 1+2x+2x^2+x^3,\\ p_7(x) &= 1+(1+\alpha)x+\alpha(1+\alpha)x^2+(1+\alpha)x^3+x^4, \quad \alpha=2\sin\left(\tfrac{3\pi}{14}\right),\\ p_8(x) &= 1+(1+\sqrt{2})x+(2+\sqrt{2})x^2+(2+\sqrt{2})x^3+(1+\sqrt{2})x^4+x^5. \end{aligned}$$

In each case, positivity of the coefficients, together with the fact that they are fixed under $\mathcal{L}$, shows that they are ∞-log-concave. None of these polynomials has all their roots on the negative real axis. Hence, it is not possible to deduce the ∞-log-concavity of their coefficients using Brändén's result in form of Corollary 2.3.

Remark 3.9. Note that $1/\cos(2\pi/s)\to 1$ as $s\to\infty$. Part (e) of Theorem 3.7 therefore shows that, given any $\varepsilon>0$, there is a finite sequence (a_n) which is ∞-log-concave but not $(1+\varepsilon)$-factor log-concave.

In particular, let (a_n) be the coefficients of the polynomial $p_s(x)$ with $s\geqslant 6$. The above shows that (a_n) is fixed by $\mathcal{L}$ but is not r-factor log-concave for $r>2$. In particular, there is no $m>0$ such that $\mathcal{L}^m(a_n)$ is r-factor log-concave for $r>2$. It is therefore not possible to apply Lemma 2.5 as in Example 2.6 to show that the sequence (a_n) is ∞-log-concave.

Apart from the special case discussed in Remark 3.9, we have been able to successfully apply the approach of Example 2.6 to establish ∞-log-concavity in all the examples we have encountered. This motivates the next question which, in particular, asks whether ∞-log-concavity of a finite sequence is decidable.

Question 3.10. Given a finite positive sequence (a_n), compute $\mathcal{L}^m(a_n)$ for $m=1,2,\ldots$ until either

(a) $\mathcal{L}^m(a_n)$ has negative terms, or
(b) $\mathcal{L}^m(a_n)=\lambda(a_n)$ for some $\lambda>0$, or

(c) $\mathcal{L}^m(a_n)$ is r-factor log-concave for some $r \geqslant \frac{3+\sqrt{5}}{2} \approx 2.618$.

In the case (a) the sequence is $(m-1)$-log-concave but not m-log-concave, and in the cases (b) and (c) the sequence is ∞-log-concave.

Does this simple algorithm always terminate? Or, if this is not the case, is there some other algorithm which determines, in finite time, whether a given finite sequence is ∞-log-concave?

4. Sequences fixed by $\mathcal{L}^2$

We now consider an analog of Theorem 3.3 which characterizes sequences fixed by $\mathcal{L}^2$. No such characterization appears to exist for sequences fixed by $\mathcal{L}^n$ with $n > 2$; see Example 4.3. Note that the rational generating function in Theorem 4.1 is equivalent to the sequence (a_n) satisfying the recurrence

$$a_n - \beta a_{n-1} + (\beta^2 - \gamma)a_{n-2} - a_{n-3} = 0,$$

with initial conditions $a_0 = 1$ and $a_{-1} = a_{-2} = a_{-3} = 0$.

Theorem 4.1. *Let β, γ be arbitrary numbers. Then the sequence $(a_n)_{n\geqslant 0}$ defined by*

$$\sum_{n\geqslant 0} a_n x^n = \frac{1}{1 - \beta x + (\beta^2 - \gamma)x^2 - x^3}$$

is fixed by $\mathcal{L}^2$. Note that $a_0 = 1$, $a_1 = \beta$, $a_2 = \gamma$.

Proof. Let $\alpha_1, \alpha_2, \alpha_3$ be such that

$$1 - \beta x + (\beta^2 - \gamma)x^2 - x^3 = (\alpha_1 - x)(\alpha_2 - x)(\alpha_3 - x).$$

Necessarily, $\alpha_1\alpha_2\alpha_3 = 1$. In the sequel, we assume that $\alpha_1, \alpha_2, \alpha_3$ are distinct; the general case may then be obtained by a limiting argument. Using the general partial fraction expansion

$$\prod_{j=1}^{d} \frac{1}{\alpha_j - x} = \sum_{k=1}^{d} \frac{1}{\alpha_k - x} \prod_{\substack{j=1\\ j\neq k}}^{d} \frac{1}{\alpha_j - \alpha_k},$$

valid for distinct α_j, we obtain the expansion

$$\prod_{j=1}^{d} \frac{1}{\alpha_j - x} = \sum_{n=0}^{\infty} x^n \sum_{k=1}^{d} \frac{1}{\alpha_k^{n+1}} \prod_{\substack{j=1\\ j\neq k}}^{d} \frac{1}{\alpha_j - \alpha_k}.$$

In the case $d = 3$, we thus have

$$a_n = \frac{1}{(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)} \left[\frac{\alpha_3 - \alpha_2}{\alpha_1^{n+1}} + \frac{\alpha_1 - \alpha_3}{\alpha_2^{n+1}} + \frac{\alpha_2 - \alpha_1}{\alpha_3^{n+1}} \right]. \tag{10}$$

By a straight-forward direct computation, using $\alpha_1\alpha_2\alpha_3 = 1$, we find that

$$a_n^2 - a_{n-1}a_{n+1} = \frac{1}{(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)} \left[\frac{\alpha_3 - \alpha_2}{\alpha_1^{-n-2}} + \frac{\alpha_1 - \alpha_3}{\alpha_2^{-n-2}} + \frac{\alpha_2 - \alpha_1}{\alpha_3^{-n-2}} \right].$$

Comparing this expression for $\mathcal{L}(a_n)$ with (10) for (a_n), it becomes clear that iterating this computation to compute $\mathcal{L}^2(a_n)$ will result in the original sequence. In other words, we have shown that $\mathcal{L}^2(a_n) = (a_n)$, as claimed.

An alternative and automatic, but arguably less illuminating and rather more computational, proof can be obtained by applying the C-finite ansatz [19] as indicated in Remark 3.4. □

Example 4.2. If $\beta = 2$ and $\gamma = 3$, then we obtain the sequence (a_n) whose first few terms are given by

$$1, 2, 3, 5, 9, 16, 28, 49, 86, 151, 265, \ldots$$

Indeed, this sequence [16, A005314] may be written as

$$a_{n-1} = \sum_{k=0}^{\lfloor (n-1)/3 \rfloor} \binom{n-k}{2k+1},$$

and a_{n-1} counts the number of compositions of n into parts congruent to 1 or 2 modulo 4. Theorem 4.1 shows that this sequence is fixed by $\mathcal{L}^2$. However, $\mathcal{L}(a_n) = (1, 1, -1, -2, 1, 4, \ldots)$ is not positive, so (a_n) is not even log-concave.

Example 4.3. After the success of Theorems 3.3 and 4.1, one may be tempted to hope that similar characterizations exist for fixed sequences of $\mathcal{L}^m$ when $m > 2$. This does not seem to be the case, as is illustrated by the following example for $m = 3$. The sequence (a_n), characterized by $a_0 = 1$, $a_1 = 2$, $a_2 = 5$, $a_3 = 9$ together with the fact that it is fixed by $\mathcal{L}^3$, is given by

$$1, 2, 5, 9, \frac{96}{5}, \frac{324547}{9450}, \frac{4079971657981}{58296672000}, \ldots$$

Computing several more terms, one can check that (a_n) cannot satisfy a linear recurrence of small degree and small order. For instance, we have found that it does not satisfy a recurrence with constant coefficients of order up to 10 (recall that in the cases $m = 1$ and $m = 2$ any such sequence satisfies a recurrence with constant coefficients of order 3). We also found that (a_n) does not satisfy a recurrence with linear coefficients of order up to 6, or a recurrence with quadratic coefficients of order up to 4.

In all the examples we have considered of sequences that are fixed by $\mathcal{L}^3$ with initial integral values as in Example 4.3, the resulting sequences involved fractions of rapidly increasing size. We are therefore lead to wonder whether there are examples of integer sequences fixed by $\mathcal{L}^3$ (but not by $\mathcal{L}$).

Question 4.4. For $n > 2$, are there (positive) integer sequences that are fixed by $\mathcal{L}^n$, but not by $\mathcal{L}^m$ for any $1 \leqslant m < n$?

5. Convolutions of sequences

It is a well-known result of Hoggar [9] that the product of polynomials with positive and log-concave coefficients again has log-concave coefficients. This is generalized in [10] by Johnson and Goldschmidt who apply their result to show, for instance, log-concavity of the Stirling numbers of the second kind as a sequence in the second parameter.

This naturally leads one to wonder whether anything interesting can be said about the log-concavity of the coefficients of the product $p(x) \cdot q(x)$ if $p(x)$ and $q(x)$ have m- and n-log-concave coefficients, respectively. The next example, constructed as an application of the previous sections, dampens excessive expectations by demonstrating that the product of two polynomials with ∞-log-concave coefficients may fail to be 5-log-concave.

Example 5.1. Consider the polynomial

$$p(x) = 1 + (1+\sqrt{2})x + (2+\sqrt{2})x^2 + (2+\sqrt{2})x^3 + (1+\sqrt{2})x^4 + x^5 \tag{11}$$

from Example 3.8. Since it is fixed under $\mathcal{L}$, its coefficients are ∞-log-concave. However, the coefficients of $p(x)^2$ are 4-log-concave but not 5-log-concave. On the other hand, the coefficients of $p(x)^n$ for $n = 3, 4, \ldots, 50$ are again ∞-log-concave.

We remark that slight perturbations of the polynomial $p(x)$ in (11) can be used to construct many further examples of polynomials with similar properties but which are not fixed under $\mathcal{L}$. For instance, the degree 6 polynomial $q(x) = p(x) + (x/4)^6$ has the property that its coefficients are 5-log-concave, while the coefficients of $q(x)^2$ are only 4-log-concave but not 5-log-concave.

Note that Example 5.1 demonstrates that, in general, if the coefficients of $p(x)^\lambda$ are m-log-concave for some $m > 1$ this does not imply that the coefficients of $p(x)^{\lambda+1}$ are m-log-concave as well. On the other hand, note that each polynomial $p(x) = a_0 + a_1 x + \cdots + a_d x^d$ with positive coefficients $a_j > 0$, $j = 0, 1, \ldots, d$, induces a probability distribution on the set $\{0, 1, \ldots, d\}$ with probability weights specified by the coefficients of $p(x)$. Namely, a random variable X is distributed according to $p(x)$, if, for $j = 0, 1, \ldots, d$,

$$\mathrm{Prob}(X = j) = \frac{a_j}{p(1)}.$$

If X and Y are random variables distributed according to $p(x)$ and $q(x)$, respectively, then $X + Y$ is distributed according to the product $p(x)q(x)$. In particular, let $X_1, \ldots, X_\lambda$ be independent random variables distributed according to $p(x)$. Then their sum $X_1 + \cdots + X_\lambda$ is distributed according to $p(x)^\lambda$. The central limit theorem therefore suggests that the coefficients of $p(x)^\lambda$ should eventually become "more log-concave" as λ increases. Indeed, in all the examples we have considered, we have observed that the coefficients of $p(x)^\lambda$ become ∞-log-concave for finite λ.

Conjecture 5.2. *Let $p(x)$ be a polynomial with positive coefficients, that is $p(x) = a_0 + a_1 x + \cdots + a_d x^d$ with $a_j > 0$ for all $j = 0, 1 \ldots, d$. Then there exists N such that, for all $\lambda \geqslant N$, $p(x)^\lambda$ has coefficients that are ∞-log-concave.*

Example 5.3. Consider $p(x) = 1+x+x^2$. Computer experiments suggest that the coefficients of p^λ are ∞-log-concave for $\lambda \geqslant 10$. For $10 \leqslant \lambda \leqslant 500$ we have proved this using Lemma 2.5 as in Example 2.6. For instance, in the case $\lambda = 10$, one observes that the coefficients of $\mathcal{L}^5[p^{10}]$ are 9.10-factor log-concave. More generally, it appears that p^λ is log-concave for $\lambda \geqslant 1$ (and strongly log-concave for $\lambda \geqslant 2$), 2-log-concave for $\lambda \geqslant 4$, 3-log-concave for $\lambda \geqslant 7$, 4-log-concave for $\lambda \geqslant 8$, 5-log-concave for $\lambda \geqslant 9$, and, as mentioned above, ∞-log-concave for $\lambda \geqslant 10$. Note that this information is included in Table 1.

Example 5.4. Consider the polynomial $p(x) = 1 + x + 2x^2$ and note that the coefficients $(1, 1, 2)$ are not log-concave. However, p^3 has log-concave coefficients, p^{10} has 2-log-concave and p^{16} has 3-log-concave coefficients. This information is further extended in Table 1. In particular, it turns out that p^{23} has ∞-log-concave coefficients. Indeed, the coefficients of $\mathcal{L}^5[p^{23}]$ are 4.23-factor log-concave.

Example 5.5. Let us consider general quadratic polynomials $p(x) = a_0 + a_1x + a_2x^2$. Note that scaling a polynomial $p(x) \to \lambda p(x)$ does not affect questions of log-concavity, and neither does the transformation $p(x) \to p(\lambda x)$. Without loss of generality, we may therefore assume $a_0 = 1$ and $a_1 = 1$. In the case of the polynomials $p(x) = 1 + x + ax^2$, $a \in \{1, 2, \ldots, 6\}$, and values $m \in \{1, 2, \ldots, 10, \infty\}$, Table 1 lists the minimal exponent λ such that p^λ has m-log-concave coefficients.

	1	2	3	4	5	6	7	8	9	10	∞
$1 + x + x^2$	1	4	7	8	9	10	10	10	10	10	10
$1 + x + 2x^2$	3	10	16	20	21	22	23	23	23	23	23
$1 + x + 3x^2$	5	16	26	31	33	35	36	36	36	36	36
$1 + x + 4x^2$	7	22	35	42	46	48	49	49	49	49	49
$1 + x + 5x^2$	9	28	45	53	58	61	61	62	62	62	62
$1 + x + 6x^2$	11	34	54	65	70	73	74	75	75	75	75

TABLE 1. Minimal exponent λ such that p^λ has m-log-concave coefficients.

We invite the reader to observe the various loose patterns suggested by the data contained in Table 1.

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