# Densities of short uniform random walks in higher dimensions

Jonathan M. Borwein[1], Armin Straub[2], and Christophe Vignat[3]

[1]University of Newcastle
[2]University of Illinois at Urbana-Champaign
[3]Tulane University

April 3, 2015

**Abstract**

We study arithmetic properties of short uniform random walks in arbitrary dimensions, with a focus on explicit (hypergeometric) evaluations of the moment functions and probability densities in the case of up to five steps. Somewhat to our surprise, we are able to provide complete extensions to arbitrary dimensions for most of the central results known in the two-dimensional case.

## 1 Introduction

An $n$-step uniform random walk in $\mathbb{R}^d$ starts at the origin and consists of $n$ independent steps of length 1, each of which is taken into a uniformly random direction. In other words, each step corresponds to a random vector uniformly distributed on the unit sphere. The study of such walks originates with Pearson [Pea05], who was interested in planar walks, which he looked at [Pea06] as migrations of, for instance, mosquitos moving a step after each breeding cycle. Random walks in three dimensions (known as random flights) seem first to have been studied in extenso by Rayleigh [Ray19], and higher dimensions are touched upon in [Wat41, §13.48].

This paper is a companion to [BNSW11, BSW13] and [BSWZ12], which studied the analytic and number theoretic behaviour of short uniform random walks in the plane (five steps or less). In this work we revisit the same issues in higher dimensions. Somewhat to our surprise, we are able to provide complete extensions for most of the central results in the culminating paper [BSWZ12].

Throughout the paper, $n$ and $d$ are the positive integers corresponding to the number of steps and the dimension of the random walk we are considering. Moreover, we denote with $\nu$ the half-integer

$$\nu = \frac{d}{2} - 1. \tag{1}$$

It turns out that most results are more naturally expressed in terms of this parameter $\nu$, and so we denote, for instance, with $p_n(\nu; x)$ the probability density function of the distance to the origin after

2010 *Mathematics subject classification.* Primary 33C20, 60G50; Secondary 05A19

*Key words and phrases.* short random walks, generalized hypergeometric functions, Bessel integrals, Narayana numbers.

$n$ random unit steps in $\mathbb{R}^d$. In Section 2.1, we mostly follow the account in [Hug95] and develop the basic Bessel integral representations for these densities beginning with Theorems 2.1 and 2.10, which are central to our analysis. In particular, a brief discussion of the (elementary) case of odd dimensions is included in Section 2.2.

In Section 2.3, we turn to general results on the associated moment functions

$$W_n(\nu; s) = \int_0^\infty x^s p_n(\nu; x)\mathrm{d}x. \tag{2}$$

In particular, we derive in Theorem 2.18 a formula for the even moments $W_n(\nu; 2k)$ as a multiple sum over the product of multinomial coefficients. As a consequence, we observe another interpretation of the Catalan numbers as the even moments of the distance after two random steps in four dimensions, and realize, more generally, in Example 2.22 the moments in four dimensions in terms of powers of the Narayana triangular matrix. We shall see that dimensions two and four are privileged in that all even moments are integral only in those two dimensions.

In Section 3, we turn to a detailed analysis of the moments of short step walks: from two-step walks (§3.1) through five steps (§3.4). For instance, we show in (37), Theorem 3.6 and Example 3.25 that the ordinary generating functions of the even moments for $n = 2, 3, 4$ can be expressed in terms of hypergeometric functions. We are also able to give closed forms for all odd moments for less than five steps.

In Section 4, we perform a corresponding analysis of the densities $p_n(\nu; x)$: from two-step walks (§4.1) through five steps (§4.4). One especially striking result for $n = 3$, shown in Corollary 4.6, is the following functional equation for the probability density function $p_3(\nu; x)$. For $0 \leqslant x \leqslant 3$, and each half-integer $\nu \geqslant 0$, the function $F(x) := p_3(\nu; x)/x$ satisfies the functional equation

$$F(x) = \left(\frac{1+x}{2}\right)^{6\nu-2} F\left(\frac{3-x}{1+x}\right). \tag{3}$$

Finally, in Section 5, we make some concluding remarks and leave several open questions.

As much as possible, we keep our notation consistent with that in [BNSW11, BSW13], and especially [BSWZ12], to which we refer for details of how to exploit the Mellin transform and similar matters. Random walks in higher dimensions are also briefly discussed in [Wan13, Chapter 4]. In particular, Wan gives evaluations in arbitrary dimensions for the second moments, which we consider and generalize in Example 2.20, as well as for two-step walks.

# 2 Basic results from probability

## 2.1 The probability densities

For the benefit of the reader, we briefly summarize the account given in [Hug95, Chapter 2.2] of how to determine the probability density function $p_n(\nu; x)$ of an $n$-step random walk in $d$ dimensions. The reader interested in further details and corresponding results for more general random walks, for instance, with varying step sizes, is referred to [Hug95].

Throughout the paper, the *normalized Bessel function of the first kind* is

$$j_\nu(x) = \nu! \left(\frac{2}{x}\right)^\nu J_\nu(x) = \nu! \sum_{m \geqslant 0} \frac{(-x^2/4)^m}{m!(m+\nu)!}. \tag{4}$$

With this normalization, we have $j_\nu(0) = 1$ and

$$j_\nu(x) \sim \frac{\nu!}{\sqrt{\pi}} \left(\frac{2}{x}\right)^{\nu+1/2} \cos\left(x - \frac{\pi}{2}\left(\nu + \frac{1}{2}\right)\right) \tag{5}$$

as $x \to \infty$ on the real line. Note also that $j_{1/2}(x) = \operatorname{sinc}(x) = \sin(x)/x$, which in part explains why analysis in 3-space is so simple. More generally, all half-integer order $j_\nu(x)$ are elementary.

**Theorem 2.1. (Bessel integral for the densities, I)** *The probability density function of the distance to the origin in $d \geqslant 2$ dimensions after $n \geqslant 2$ steps is, for $x > 0$,*

$$p_n(\nu; x) = \frac{2^{-\nu}}{\nu!} \int_0^\infty (tx)^{\nu+1} J_\nu(tx) j_\nu^n(t) \mathrm{d}t, \tag{6}$$

*where, as introduced in (1), $\nu = \frac{d}{2} - 1$.*

*Proof.* Let $\boldsymbol{X}$ be a random vector, which is uniformly distributed on the unit sphere in $\mathbb{R}^d$. That is, $\boldsymbol{X}$ describes the displacement of a single step in our random walk. Then the Fourier transform of its induced probability measure $\mu_{\boldsymbol{X}} = \frac{\Gamma(d/2)}{2\pi^{d/2}} \delta\left(\|\boldsymbol{x}\| - 1\right)$ is

$$\hat{\mu}_{\boldsymbol{X}}(\boldsymbol{q}) = \int_{\mathbb{R}^d} e^{i\langle \boldsymbol{x}, \boldsymbol{q}\rangle} \mathrm{d}\mu_{\boldsymbol{X}}(\boldsymbol{x}) = j_\nu(\|\boldsymbol{q}\|).$$

This is a special case of the famous formula [Hug95, (2.30)],

$$\int_{\mathbb{R}^d} e^{i\langle \boldsymbol{x}, \boldsymbol{q}\rangle} f(\|\boldsymbol{x}\|) \mathrm{d}\boldsymbol{x} = 2\pi^{d/2} \int_0^\infty \left(\frac{2}{tq}\right)^{d/2-1} J_{d/2-1}(tq) t^{d-1} f(t) \mathrm{d}t, \tag{7}$$

with $q = \|\boldsymbol{q}\|$, for integrals of orthogonally invariant functions.

Note that the position $\boldsymbol{Z}$ of a random walk after $n$ unit steps in $\mathbb{R}^d$ is distributed like the sum of $n$ independent copies of $\boldsymbol{X}$. The Fourier transform of $\mu_{\boldsymbol{Z}}$ therefore is

$$\hat{\mu}_{\boldsymbol{Z}}(\boldsymbol{q}) = j_\nu^n(\|\boldsymbol{q}\|). \tag{8}$$

We are now able to obtain the probability density function $p_n(\nu; \boldsymbol{x})$ of the position after $n$ unit steps in $\mathbb{R}^d$ via the inversion relation

$$p_n(\nu; \boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \boldsymbol{x}, \boldsymbol{q}\rangle} \hat{\mu}_{\boldsymbol{Z}}(\boldsymbol{q}) \mathrm{d}\boldsymbol{q}. \tag{9}$$

Combining (8), (9) and (7), we find

$$p_n(\nu; \boldsymbol{x}) = \frac{1}{(2\pi)^{\nu+1}} \int_0^\infty \frac{t^{\nu+1}}{x^\nu} J_\nu(tx) j_\nu^n(t) \mathrm{d}t.$$

Since the surface area of the unit sphere in $\mathbb{R}^d$ is $2\pi^{\nu+1}/\nu!$, the density functions for the position and distance are related by

$$p_n(\nu; \boldsymbol{x}) = \frac{\nu!}{2\pi^{\nu+1} \|\boldsymbol{x}\|^{2\nu+1}} p_n(\nu; \|\boldsymbol{x}\|), \tag{10}$$

whence we arrive at the formula (6). ∎

The probability densities $p_3(\nu; x)$ of the distance to the origin after three random steps in dimensions $2, 3, \ldots, 9$ are depicted in Figure 1. With the exception of the planar case, which has a logarithmic singularity at $x = 1$, these functions are at least continuous in the interval $[0, 3]$, on which they are supported. Their precise regularity is provided by Corollary 2.5. For comparison, the probability densities of four-step walks in dimensions $2, 3, \ldots, 9$ are plotted in Figure 2, and corresponding plots for five steps are provided by Figure 4.

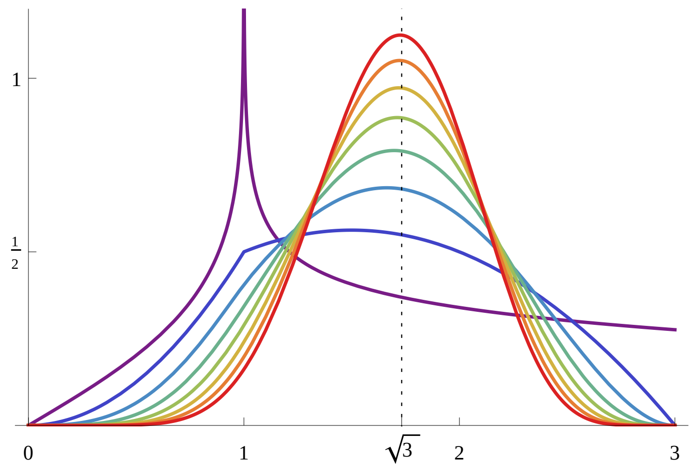


Figure 1: $p_3(\nu; x)$ for $\nu = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}$

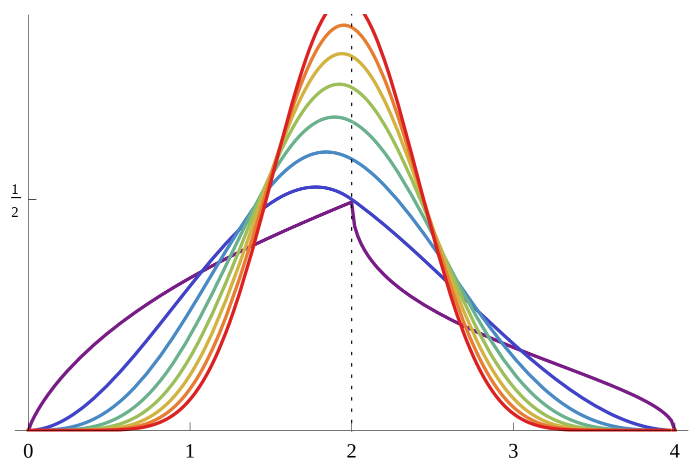


Figure 2: $p_4(\nu; x)$ for $\nu = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}$

Observe how the density functions center and spike as the dimensions increase. Indeed, this is a general phenomenon and the distributions described by the densities $p_n(\nu; x)$ approach a Dirac distribution centered at $\sqrt{n}$. The intuition behind this observation is as follows: as the dimension $d$ increases, the directions of each of the $n$ random steps increasingly tend to be close to orthogonal to each other. That is, given an incoming direction, the direction of the next step will probably belong to a hyperplane which is almost orthogonal to this direction. Pythagoras' theorem therefore predicts that the distance after $n$ steps is about $\sqrt{n}$. A precise asymptotic result, which confirms this intuition, is given in Example 2.4.

Integrating the Bessel integral representation (6) for the probability density functions $p_n(\nu; x)$, we obtain a corresponding Bessel integral representation for the cumulative distribution functions,

$$P_n(\nu; x) = \int_0^x p_n(\nu; y) \mathrm{d}y, \tag{11}$$

of the distance to the origin after $n$ steps in $d$ dimensions.

**Corollary 2.2. (Cumulative distribution)** *Suppose $d \geqslant 2$ and $n \geqslant 2$. Then, for $x > 0$,*

$$P_n(\nu; x) = \frac{2^{-\nu}}{\nu!} \int_0^\infty (tx)^{\nu+1} J_{\nu+1}(tx) j_\nu^n(t) \frac{\mathrm{d}t}{t}. \tag{12}$$

**Example 2.3. (Kluyver's Theorem)** A famous result of Kluyver [Klu06] asserts that, for $n \geqslant 2$,

$$P_n(0; 1) = \frac{1}{n+1}.$$

That is, after $n$ unit steps in the plane, the probability to be within one unit of the starting point is $1/(n+1)$. This is nearly immediate from (12). An elementary proof of this remarkable result was given recently by Bernardi [Ber13].

It is natural to wonder whether there exists an extension of this result to higher dimensions. Clearly, these probabilities decrease as the dimension increases.

(a) In the case of two steps, we have

$$P_2(\nu; x) = \frac{x^{2\nu+1}}{2\sqrt{\pi}} \frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)} {}_2F_1\left( \left. \begin{matrix} \frac{1}{2}+\nu, \frac{1}{2}-\nu \\ \frac{3}{2}+\nu \end{matrix} \right| \frac{x^2}{4} \right),$$

which, in the case of integers $\nu \geqslant 0$ and $x = 1$, reduces to

$$P_2(\nu; 1) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \sum_{k=0}^{\nu-1} \frac{3^k}{(2k+1)\binom{2k}{k}}. \tag{13}$$

Alternatively,

$$P_2(\nu; 1) = \frac{1}{3} - \frac{1}{4\pi} \sum_{k=1}^{\nu} 3^{k-1/2} \frac{\Gamma(k)^2}{\Gamma(2k)}. \tag{14}$$

In particular, in dimensions 4, 6 and 8,

$$P_2(1; 1) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi}, \quad P_2(2; 1) = \frac{1}{3} - \frac{3\sqrt{3}}{8\pi}, \quad P_2(3; 1) = \frac{1}{3} - \frac{9\sqrt{3}}{20\pi}, \quad \ldots,$$

and it is obvious from (13) that all the probabilities $P_2(\nu; 1)$ are of the form $\frac{1}{3} - c_\nu \frac{\sqrt{3}}{\pi}$ for some rational factor $c_\nu > 0$.

(b) In the case of three steps, we find

$$P_3(1;1) = \frac{1}{4} - \frac{4}{3\pi^2}, \quad P_3(2;1) = \frac{1}{4} - \frac{256}{135\pi^2}, \quad P_3(3;1) = \frac{1}{4} - \frac{2048}{945\pi^2}, \quad \ldots$$

Indeed, for integers $\nu \geqslant 0$, we have the general formula

$$P_3(\nu;1) = \frac{1}{4} - \frac{1}{3\pi^2} \sum_{k=1}^{\nu} 2^{6(k-1)}(11k-3)\frac{\Gamma^5(k)}{\Gamma(2k)\Gamma(3k)}. \tag{15}$$

In the limit we arrive at the improbable evaluation

$${}_5F_4\left(\left.\begin{matrix} \frac{19}{11}, 1, 1, 1, 1 \\ \frac{8}{11}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3} \end{matrix} \right| \frac{16}{27}\right) = \frac{3}{16}\pi^2, \tag{16}$$

since $P_3(\nu;1)$ goes to zero as the dimension goes to infinity.

The case of $n$-step walks, with $n \geqslant 4$, is much less accessible [BBBG08, §5], and it would be interesting to obtain a more complete extension of Kluyver's result to higher dimensions. ◇

**Example 2.4.** Asymptotically, for $x > 0$ as $\nu \to \infty$,

$$j_\nu(t) \sim \exp\left(-\frac{t^2}{4\nu+2}\right).$$

It may thus be derived from the Bessel integral representation (6) that asymptotically, as the dimension goes to infinity, $p_n(\nu;x) \sim q_n(\nu;x)$, where

$$q_n(\nu;x) = \frac{2^{-\nu}}{\nu!}\left(\frac{2\nu+1}{n}\right)^{\nu+1} x^{2\nu+1} \exp\left(-\frac{2\nu+1}{2n}x^2\right)$$

for $x \geqslant 0$ and $q_n(\nu;x) = 0$ for $x < 0$. The probability density $q_n(\nu;x)$ describes a scaled chi distribution with $2\nu + 2 = d$ degrees of freedom. Its average is

$$\int_0^\infty x q_n(\nu;x)\mathrm{d}x = \sqrt{\frac{2n}{2\nu+1}}\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{\Gamma(\nu+1)},$$

which converges to $\sqrt{n}$ as $\nu \to \infty$. More generally, for the $s$th moment, with $s > -1$,

$$\begin{aligned} \int_0^\infty x^s q_n(\nu;x)\mathrm{d}x &= \left(\frac{2n}{2\nu+1}\right)^{s/2} \frac{\Gamma\left(\nu+\frac{s}{2}+1\right)}{\Gamma(\nu+1)} \\ &\sim \left(\frac{2n\nu}{2\nu+1}\right)^{s/2}\left(1 + \frac{s(s+2)}{8\nu} + O(\nu^{-2})\right), \end{aligned}$$

as $\nu \to \infty$, and it is straightforward to compute further terms of this asymptotic expansion. The fact that the $s$th moment approaches $n^{s/2}$ for large dimensions, of course, reflects the observation from Figures 1, 2 and 4 that the probability densities $p_n(\nu;x)$ center and spike at approximately $\sqrt{n}$. ◇

These Bessel integral representations also allow us to deduce the regularity of the density functions.

**Corollary 2.5. (Regularity of the density)** *The density $p_n(\nu; x)$ is $m$ times continuously differentiable for $x > 0$ if*

$$n > \frac{m+1}{\nu + 1/2} + 1, \quad \text{or, equivalently,} \quad m < (n-1)(\nu + 1/2) - 1.$$

*Proof.* It follows from (6), that

$$p_n(\nu; x) = \frac{2^{-2\nu}}{\nu!^2} \int_0^\infty (tx)^{2\nu+1} j_\nu(tx) j_\nu^n(t) \mathrm{d}t. \tag{17}$$

Observe that this integral converges absolutely if $2\nu + 1 - (n+1)(\nu + 1/2) < -1$, in which case $p_n(\nu; x)$ is continuous. Repeatedly differentiating under the integral sign as long as is permitted, we conclude that $p_n(\nu; x)$ is $m$ times continuously differentiable if

$$2\nu + 1 - (n+1)(\nu + 1/2) < -1 - m,$$

and it only remains to solve for $n$, respectively $m$. □

**Example 2.6.** In the case $d = 2$, this implies that $p_n(0; x)$ is in $C^0$ for $n > 3$, in $C^1$ for $n > 5$, in $C^2$ for $n > 7$, and so on. Indeed, $p_3(0; x)$ has a logarithmic singularity at $x = 1$, and $p_4(0; x)$ as well as $p_5(0; x)$ are not differentiable at $x = 2, 4$ and $x = 1, 3, 5$, respectively. ◇

**Corollary 2.7.** *Let $n \geqslant 4$ and $d \geqslant 2$ such that $(n, d) \neq (4, 2)$. Then,*

$$\frac{1}{(2\nu+1)!} p_n^{(2\nu+1)}(\nu; 0) = p_{n-1}(\nu; 1), \tag{18}$$

*where the derivative is understood to be taken from the right.*

*Proof.* Starting with (17), we differentiate $2\nu + 1$ many times, and compare with

$$p_{n-1}(\nu; 1) = \frac{2^{-2\nu}}{\nu!^2} \int_0^\infty t^{2\nu+1} j_\nu^n(t) \mathrm{d}t.$$

□

In the case $d = 2$, this reduces to $p_n'(0; 0) = p_{n-1}(0; 1)$, which was crucial in [BSWZ12] for explicitly evaluating $p_5'(0; 0)$.

**Example 2.8.** As long as $p_n(\nu; x)$ is sufficiently differentiable at $x = 1$, we can further relate the value $p_n(\nu; 1)$, occuring in (18), to corresponding values of derivatives. Let us illustrate this by showing that

$$p_n'(\nu; 1) = \frac{2n\nu + n - 1}{n+1} p_n(\nu; 1) \tag{19}$$

for all $n \geqslant 3$, $\nu > 0$ such that $(n, \nu) \neq (3, 1/2)$. In that case, we may, as in Corollary 2.5, differentiate (17) under the integral sign to obtain

$$p_n'(\nu; 1) = (2\nu + 1) p_n(\nu; 1) + \frac{2^{-2\nu}}{\nu!^2} \int_0^\infty t^{2\nu+2} j_\nu'(t) j_\nu^n(t) \mathrm{d}t$$

On the other hand, integrating (17) by parts, we find

$$p_n(\nu;1) = -\frac{n+1}{2\nu+2}\frac{2^{-2\nu}}{\nu!^2}\int_0^\infty t^{2\nu+2} j_\nu^n(t) j_\nu'(t)\mathrm{d}t.$$

Combining these, we arrive at (19). ◇

The densities of an $n$-step walk can be related to the densities of an $(n-1)$-step walk by the following generalization of [Hug95, (2.70)].

**Theorem 2.9. (Recursion for the density)** *For $x>0$ and $n=1,2,\ldots,$ the function*

$$\psi_n(\nu;x) = \frac{\nu!}{2\pi^{\nu+1}}\frac{p_n(\nu;x)}{x^{2\nu+1}}$$

*satisfies*

$$\psi_n(\nu;x) = \frac{\nu!^2 2^{2\nu}}{(2\nu)!\pi}\int_{-1}^1 \psi_{n-1}(\nu;\sqrt{1+2\lambda x+x^2})(1-\lambda^2)^{\nu-1/2}\mathrm{d}\lambda. \tag{20}$$

*Proof.* Recall from (10) that the probability density of the position after $n$ steps in $\mathbb{R}^d$ is

$$p_n(\nu;\boldsymbol{x}) = \psi_n(\nu;\|\boldsymbol{x}\|).$$

Since the steps of our walks are uniformly distributed vectors on the unit sphere in $\mathbb{R}^d$, we have

$$p_n(\nu;\boldsymbol{x}) = \int_{\|\boldsymbol{y}\|=1} p_{n-1}(\nu;\boldsymbol{x}-\boldsymbol{y})\mathrm{d}S = \int_{\|\boldsymbol{y}\|=1}\psi_{n-1}(\nu;\|\boldsymbol{x}-\boldsymbol{y}\|)\mathrm{d}S,$$

where $\mathrm{d}S$ denotes the normalized surface measure of the unit sphere. After introducing $d$-dimensional spherical polar coordinates, as detailed, for instance, in [Hug95, p. 61], and a straightforward change of variables we arrive at (20). □

Finally, a computationally more accessible expression for the densities $p_n(\nu;x)$ is given by the following generalization of a formula, which was derived by Broadhurst [Bro09] in the case of two dimensions, that is, $\nu=0$. Note that (6) is the special case $k=0$ in (21).

**Theorem 2.10. (Bessel integral for the densities, II)** *Let $n \geqslant 2$ and $d \geqslant 2$. For any nonnegative integer $k$, and $x>0$,*

$$p_n(\nu;x) = \frac{2^{-\nu}}{\nu!}\frac{1}{x^{2k}}\int_0^\infty (tx)^{\nu+k+1}J_{\nu+k}(tx)\left(-\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^k j_\nu^n(t)\mathrm{d}t. \tag{21}$$

*Proof.* As in [Bro09], we proceed by induction on $k$. The case $k=0$ is (6). Suppose that (21) is known for some $k$. By the Bessel function identity

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z) \tag{22}$$

we find that, for any smooth function $g(t)$ which is sufficiently small at 0 and $\infty$,

$$\begin{aligned}\int_0^\infty (tx)^{\alpha+1}J_\alpha(tx)g(t)\mathrm{d}t &= \frac{1}{x}\int_0^\infty\left[\frac{\mathrm{d}}{\mathrm{d}t}(tx)^{\alpha+1}J_\alpha(tx)\right]g(t)\mathrm{d}t\\ &= -\frac{1}{x}\int_0^\infty (tx)^{\alpha+1}J_\alpha(tx)\left[\frac{\mathrm{d}}{\mathrm{d}t}g(t)\right]\mathrm{d}t\\ &= \frac{1}{x^2}\int_0^\infty (tx)^{\alpha+2}J_\alpha(tx)\left[-\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}g(t)\right]\mathrm{d}t.\end{aligned}$$

In the second step, we used integration by parts and assumed that $g(t)$ is such that

$$(tx)^{\alpha+1} J_\alpha(tx) g(t) \tag{23}$$

vanishes as $t \to 0$ and $t \to \infty$. In the present case, $\alpha = \nu + k$ and $g(t) = \left(-\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^k j_\nu^n(t)$. In order to complete the proof of (21), it only remains to demonstrate that (23) indeed vanishes as required.

As $t \to 0$, we have $(tx)^{\alpha+1} J_\alpha(tx) = O(t^{2\alpha+1}) = O(t^{2\nu+2k+1})$ while $g(t) = O(1)$, because $j_\nu^n(t)$ is an even function. Hence, for $\nu > -1$, the term (23) indeed vanishes as $t \to 0$. On the other hand, as $t \to \infty$, we have $(tx)^{\alpha+1} J_\alpha(tx) = O(t^{\alpha+1/2}) = O(t^{\nu+k+1/2})$ by (5). Moreover, (5) and (22) imply that $j_\nu^{(m)}(t) = O(t^{-\nu-1/2})$, as $t \to \infty$, and therefore that $g(t) = O(t^{-n(\nu+1/2)-k})$. Since $n > 1$, it follows that (23) also vanishes as $t \to \infty$. □

We note that the $\left(-\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\right)^k j_\nu^n(t)$ in the integrand of (21) may be expressed as a finite sum of products of Bessel functions. This is made explicit in Remark 2.19.

## 2.2 The probability densities in odd dimensions

In the case of odd dimension $d$, the Bessel functions in Section 2 have half-integer index $\nu$ and are therefore elementary, so that the situation is fairly well understood since Rayleigh [Ray19, Hug95]. In particular, the probability density functions $p_n(\nu; x)$ are piecewise polynomial in odd dimensions. This is made explicit by the following theorem, which is obtained in [GP12], translated to our notation.

**Theorem 2.11. (Density in odd dimensions [GP12, Theorem 2.6])** *Assume that the dimension $d = 2m+1$ is an odd number. Then, for $0 < x < n$,*

$$p_n(m - \tfrac{1}{2}; x) = \frac{(2x)^{2m}\Gamma(m)}{\Gamma(2m)} \left(-\frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_{m,n}(x),$$

*where $P_{m,n}(x)$ is the piecewise polynomial obtained from convoluting*

$$\frac{\Gamma(m+1/2)}{\Gamma(1/2)\Gamma(m)} \begin{cases} (1-x^2)^{m-1}, & \textit{for } x \in [-1,1], \\ 0, & \textit{otherwise}, \end{cases}$$

*$n-1$ times with itself.*

**Example 2.12.** In the case $n = 3$ and $d = 3$, we have $m = 1$ and

$$P_{1,1}(x) = \frac{1}{2} \begin{cases} 1, & \text{if } |x| \leqslant 1, \\ 0, & \text{otherwise}, \end{cases}$$

as well as

$$P_{1,2}(x) = \int_{-\infty}^{\infty} P_{1,1}(t) P_{1,1}(x-t) \mathrm{d}t = \frac{1}{4} \begin{cases} 2 - |x|, & \text{if } |x| \leqslant 2, \\ 0, & \text{otherwise}, \end{cases}$$

and, hence,

$$P_{1,3}(x) = \int_{-\infty}^{\infty} P_{1,2}(t) P_{1,1}(x-t) \mathrm{d}t = \frac{1}{8} \begin{cases} 3 - x^2, & \text{if } |x| \leqslant 1, \\ \frac{1}{2}(|x|-3)^2, & \text{if } 1 < |x| \leqslant 3, \\ 0, & \text{otherwise}. \end{cases}$$

We thus find that, for $0 \leqslant x \leqslant 3$,

$$p_3(\tfrac{1}{2}; x) = 4x^2 \left( -\frac{1}{2x} \frac{\mathrm{d}}{\mathrm{d}x} \right) P_{1,3}(x) = \frac{x}{4} \begin{cases} 2x, & \text{if } 0 \leqslant x \leqslant 1, \\ 3 - x, & \text{if } 1 < x \leqslant 3. \end{cases}$$

Since $j_{1/2}(x) = \operatorname{sinc}(x) = \sin(x)/x$, evaluation of the densities in three dimensions can also be approached using the tools provided by [BB01].

Similarly, we obtain, for instance,

$$p_4(\tfrac{1}{2}; x) = \frac{x}{16} \begin{cases} x(8 - 3x), & \text{if } 0 \leqslant x \leqslant 2, \\ (4 - x)^2, & \text{if } 2 < x \leqslant 4. \end{cases} \tag{24}$$

We note that Theorem 2.11 can be usefully implemented in a computer algebra system such as *Maple* or *Mathematica*. ◇

**Example 2.13. (Moments in odd dimensions)** By integrating (24), we are able to symbolically compute the corresponding moments, as introduced in (2), as

$$W_4(1/2; s) = 2^{s+3} \frac{1 - 2^{s+2}}{(s + 2)} \frac{1}{(s + 4)(s + 3)},$$

which has a removable singularity $-2$ and poles at $-3$ and $-4$. Likewise, in five dimensions,

$$W_4(3/2; s) = \frac{(12)^3 2^{s+1} \left( s^3 + 27\, s^2 + 230\, s + 616 + 64\, 2^s \left( s^3 + 15\, s^2 + 62\, s + 56 \right) \right)}{(s + 12)(s + 10)(s + 9)(s + 8)(s + 7)(s + 6)(s + 5)(s + 4)(s + 2)}$$

with poles at $-5, -7, -8, -9, -10, -12$ and removable singularities at the other apparent poles. Thus, even this elementary evaluation has subtle structure. ◇

## 2.3 The moment functions

Theorem 2.10 allows us to prove a corresponding Bessel integral representation for the *moment function*

$$W_n(\nu; s) = \int_0^\infty x^s p_n(\nu; x) \mathrm{d}x$$

of the distance to the origin after $n$ random steps. The following result generalizes [Bro09, Theorem 1] from two to arbitrary dimensions.

**Theorem 2.14. (Bessel integral for the moments)** *Let $n \geqslant 2$ and $d \geqslant 2$. For any nonnegative integer $k$,*

$$W_n(\nu; s) = \frac{2^{s-k+1} \Gamma\left(\frac{s}{2} + \nu + 1\right)}{\Gamma(\nu + 1) \Gamma\left(k - \frac{s}{2}\right)} \int_0^\infty x^{2k-s-1} \left( -\frac{1}{x} \frac{\mathrm{d}}{\mathrm{d}x} \right)^k j_\nu^n(x) \mathrm{d}x, \tag{25}$$

*provided that $k - n(\nu + 1/2) < s < 2k$.*

*Proof.* Using Theorem 2.10, we have

$$W_n(\nu; s) = \frac{2^{-\nu}}{\nu!} \int_0^\infty x^{s-2k} \int_0^\infty (tx)^{\nu+k+1} J_{\nu+k}(tx) \left( -\frac{1}{t} \frac{\mathrm{d}}{\mathrm{d}t} \right)^k j_\nu^n(t) \mathrm{d}t \mathrm{d}x.$$

Interchanging the order of integration and substituting $z = tx$, we obtain

$$W_n(\nu; s) = \frac{2^{-\nu}}{\nu!} \int_0^\infty t^{2k-s-1} \left[ \left( -\frac{1}{t} \frac{\mathrm{d}}{\mathrm{d}t} \right)^k j_\nu^n(t) \right] \int_0^\infty z^{\nu+s-k+1} J_{\nu+k}(z) \mathrm{d}z \mathrm{d}t.$$

The inner integral may be evaluated using the standard Bessel integral evaluation

$$\int_0^\infty z^a J_\nu(z) \mathrm{d}z = \frac{2^a \Gamma\left(\frac{1+a+\nu}{2}\right)}{\Gamma\left(\frac{1-a+\nu}{2}\right)},$$

which holds for $a$ and $\nu$ such that $a + \nu > -1$ and $a < 1/2$. We conclude that

$$W_n(\nu; s) \quad = \quad \frac{2^{-\nu}}{\nu!} \int_0^\infty t^{2k-s-1} \left[ \left( -\frac{1}{t} \frac{\mathrm{d}}{\mathrm{d}t} \right)^k j_\nu^n(t) \right] \frac{2^{\nu+s-k+1} \Gamma\left(\frac{s}{2} + \nu + 1\right)}{\Gamma\left(k - \frac{s}{2}\right)} \mathrm{d}t,$$

which is the desired result. For the evaluation of the intermediate Bessel integral, we assumed $s > -2\nu - 2 = -d$ and $s < k - \nu - 1/2$, and so (25) holds for all $s$ in this non-empty strip provided that the original integral converges. Using the asymptotic bounds from the proof of Theorem 2.10, we note that the integral (25) converges absolutely in the strip $k - n(\nu + 1/2) < s < 2k$. Analytic continuation therefore implies that (25) holds for all $s$ in this strip. □

We deduce the following from (25), with $k = 0$. This extends [BSW13, Proposition 2.4]. In particular, we observe that, for $n > 2$, the first pole of $W_n(\nu; s)$ occurs at $s = -(2\nu + 2) = -d$.

**Corollary 2.15. (Poles and residues of the moments)** *Let $n > 2$. In the half-plane $\operatorname{Re} s > -n(\nu + 1/2)$, the moment functions $W_n(\nu; s)$ are analytic apart from simple poles at $s = -d - 2m$ for integers $m$ such that $0 \leqslant m < \frac{n}{2}\left(\frac{d}{2} - \frac{1}{2}\right) - \frac{d}{2}$. The residues of these poles are*

$$\operatorname{Res}_{s=-d-2m} W_n(\nu; s) = \frac{2^{-2\nu-2m}}{\nu!(\nu+m)!} \frac{(-1)^m}{m!} \int_0^\infty x^{2\nu+2m+1} j_\nu^n(x) \mathrm{d}x.$$

*Proof.* By equation (25), with $k = 0$, we have

$$W_n(\nu; s) = \frac{2^{s+1} \Gamma\left(\frac{s}{2} + \nu + 1\right)}{\Gamma(\nu+1) \Gamma\left(-\frac{s}{2}\right)} \int_0^\infty x^{-s-1} j_\nu^n(x) \mathrm{d}x,$$

valid for $s$ in the strip $-n(\nu + 1/2) < s < 0$, in which the integral converges absolutely. In the region of interest, the only poles are contributed by the factor $\Gamma\left(\frac{s}{2} + \nu + 1\right)$, which, as a function in $s$, has simple poles at $s = -d - 2m$, for $m = 0, 1, 2, \ldots$, with residue $2\frac{(-1)^m}{m!}$. □

Note that the value for the residue of $W_n(\nu; s)$ at $s = -d$ agrees with

$$p_{n-1}(\nu; 1) = \frac{1}{(2\nu+1)!} p_n^{(2\nu+1)}(\nu; 0)$$

from (18) provided that these values are finite.

**Example 2.16.** The moment functions $W_3(\nu;s)$ are plotted in Figure 3 for $\nu = 0,1,2$. The first pole of $W_3(\nu;s)$ occurs at $s=-d$, is simple and has residue

$$\operatorname{Res}_{s=-d} W_3(\nu;s) = \frac{2^{-2\nu}}{\nu!^2}\int_0^\infty x^{2\nu+1} j_\nu^3(x)\mathrm{d}x = \frac{2}{\sqrt{3}\pi}\frac{3^\nu}{\binom{2\nu}{\nu}},$$

as follows from the Bessel integral formula (72). In order to record some more general properties of the pole structure of $W_3(\nu;s)$, we use the fact that $W_3(\nu;s)$ satisfies a functional equation, (48), which relates the three terms $W_3(\nu;s)$, $W_3(\nu;s+2)$ and $W_3(\nu;s+4)$. By reversing this functional equation, we find that the residues $r_k = \operatorname{Res}_{s=-d-2k} W_3(\nu;s)$ satisfy the recursion

$$\begin{aligned} & 9(k+1)(k+\nu+1)r_{k+1} \\ = \; & \tfrac{1}{2}\left(20\left(k+\tfrac{1}{2}\right)^2 - 20\left(k+\tfrac{1}{2}\right)\nu - 4\nu^2 + 1\right) r_k - (k-\nu)(k-2\nu)r_{k-1}, \end{aligned} \tag{26}$$

with $r_{-1} = 0$ and $r_0 = \frac{2}{\sqrt{3}\pi}\frac{3^\nu}{\binom{2\nu}{\nu}}$. Observe that, when $\nu = 0$, the recursion for these residues is essentially the same as the recurrence (48) for the corresponding even moments (with $u_k$ replaced by $3^{2k}r_k$). As recorded in [BSW13, Proposition 2.4], this lead to

$$\operatorname{Res}_{s=-2(k+1)} W_3(0;s) = \frac{2}{\sqrt{3}\pi}\frac{W_3(0;2k)}{3^{2k}}.$$

Define, likewise, the numbers $V_3(\nu;k)$ by

$$\operatorname{Res}_{s=-2(\nu+k+1)} W_3(\nu;s) = \frac{2}{\sqrt{3}\pi}\frac{3^\nu}{\binom{2\nu}{\nu}}\frac{V_3(\nu;k)}{3^{2k}}.$$

In analogy with (26), we find that $u_k = V_3(\nu;k)$ solves the three-term recurrence

$$\begin{aligned} & (k+1)(k+\nu+1)u_{k+1} \\ = \; & \tfrac{1}{2}\left(20\left(k+\tfrac{1}{2}\right)^2 - 20\left(k+\tfrac{1}{2}\right)\nu - 4\nu^2 + 1\right) u_k - 9(k-\nu)(k-2\nu)u_{k-1}, \end{aligned}$$

with $u_{-1} = 0$ and $u_0 = 1$. For small dimensions, initial values for $V_3(\nu;k)$ are given by

$$\begin{array}{lll} d=2 \quad (\nu=0) & : & 1, 3, 15, 93, 639, 4653, 35169, 272835, 2157759, \ldots \\ d=4 \quad (\nu=1) & : & 1, -2, -2, -6, -24, -114, -606, -3486, -21258, \ldots \\ d=6 \quad (\nu=2) & : & 1, -5, 6, 2, 6, 18, 66, 278, 1296, \ldots \\ d=8 \quad (\nu=3) & : & 1, -\frac{15}{2}, 21, -20, 0, -9, -20, -60, -210, \ldots \end{array}$$

Note the increasingly irregular behaviour of $V_3(\nu;k)$ as $d$ increases. In particular, in dimension 8, we find $V_3(3;4) = 0$, which signifies the disappearance of the perhaps expected pole of $W_3(3;s)$ at $s=-16$. ◇

**Example 2.17.** In the case $n=4$ and $\nu=1$, that is $d=4$, the moment function $W_4(1;s)$ has a simple pole at $s=-4$ with residue

$$\operatorname{Res}_{s=-4} W_4(1;s) = \frac{1}{4}\int_0^\infty x^3 j_1^4(x)\mathrm{d}x = \frac{4}{\pi^2},$$

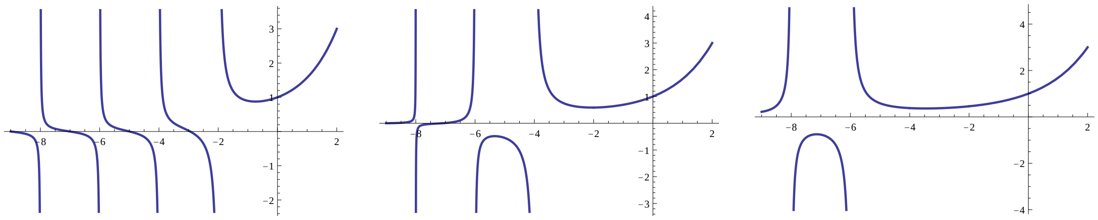

Figure 3: $W_3(\nu; s)$ on $[-9, 2]$ for $\nu = 0, 1, 2$.

and is otherwise analytic in the half-plane Re $s > -6$. At $s = -6$, on the other hand, $W_4(1; s)$ has a double pole. Indeed, analyzing the functional equations that arise from (63) and (65), we derive that

$$\lim_{s\to-6}(s+6)^2 W_4(1;s) = -\frac{1}{2}\lim_{s\to-2}(s+2)^2 W_4(0;s) = -\frac{3}{4\pi^2},$$

where in the last equality we used the known value from the planar case [BSWZ12, Example 4.3]. Similarly, we obtain

$$\begin{aligned}\operatorname{Res}_{s=-6} W_4(1;s) &= -\frac{1}{2}\operatorname{Res}_{s=-2} W_4(0;s) + \frac{1}{24}\lim_{s\to-2}(s+2)^2 W_4(0;s)\\ &= \frac{1}{16\pi^2} - \frac{9\log(2)}{4\pi^2}.\end{aligned}$$

In the higher-dimensional case, $W_4(\nu; s)$ has poles at $s = -d - 2m$ for $m = 0, 1, 2, \ldots$, which are initially simple but turn into poles of order (up to) 2 beginning at $s = -(4\nu + 2)$. $\diamond$

The approach indicated in Examples 2.16 and 2.17 enables us to determine, at least in principle, the pole structure of the moment functions $W_n(\nu; s)$ in each case. We do not pursue such a more detailed analysis herein.

We next follow the approach of [Bro09] to obtain a summatory expression for the even moments from Theorem 2.10.

**Theorem 2.18. (Multinomial sum for the moments)** *The even moments of an $n$-step random walk in dimension $d$ are given by*

$$W_n(\nu; 2k) = \frac{(k+\nu)!\nu!^{n-1}}{(k+n\nu)!}\sum_{k_1+\cdots+k_n=k}\binom{k}{k_1,\ldots,k_n}\binom{k+n\nu}{k_1+\nu,\ldots,k_n+\nu}.$$

*Proof.* Replacing $k$ by $k+1$ in (25) and setting $s = 2k$, we obtain

$$\begin{aligned}W_n(\nu;2k) &= \frac{2^k(k+\nu)!}{\nu!}\int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k j_\nu^n(x)\mathrm{d}x\\ &= \left[\frac{(k+\nu)!}{\nu!}\left(-\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k j_\nu^n(x)\right]_{x=0}. \qquad (27)\end{aligned}$$

Observe that, at the level of formal power series, we have

$$\left[\left(-\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \sum_{m\geqslant 0} a_m\left(-\frac{x^2}{4}\right)^m\right]_{x=0} = k!a_k.$$

Recall from (4) the series

$$j_\nu(x) = \nu! \sum_{m \geqslant 0} \frac{(-x^2/4)^m}{m!(m+\nu)!},$$

to conclude that

$$W_n(\nu; 2k) = \frac{(k+\nu)!}{\nu!} \nu!^n k! \sum_{m_1+\cdots+m_n=k} \frac{1}{m_1! \cdots m_n!} \frac{1}{(m_1+\nu)! \cdots (m_n+\nu)!},$$

which is equivalent to the claimed formula. □

**Remark 2.19.** Proceeding as in the proof of Theorem 2.18, we observe that

$$\left(-\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k j_\nu(x) = \frac{\nu!}{(\nu+k)!} j_{\nu+k}(x),$$

and hence

$$\begin{aligned} &\left(-\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k j_{\nu_1}(x) \cdots j_{\nu_n}(x) \\ = &\sum_{k_1+\cdots+k_n=k} \frac{k!}{k_1! \cdots k_n!} \frac{\nu_1! \cdots \nu_n!}{(k_1+\nu_1)! \cdots (k_n+\nu_n)!} j_{\nu_1+k_1}(x) \cdots j_{\nu_n+k_n}(x). \end{aligned}$$

If applied to (27), this (finite) expansion, together with $j_\nu(0) = 1$, makes the conclusion of Theorem 2.18 apparent. In conjunction with the asymptotics of $j_\nu$, we conclude, as in the proof of Theorem 2.10, that the integrand in (21) is $O(t^{-(n-1)(\nu+1/2)})$ as $t \to \infty$, so that each additional derivative improves the order at $+\infty$ by 1 — at the expense of increasing the size of the coefficients. We note, inter alia, that (27) may, alternatively, be expressed as

$$j_\nu^n(x) = \nu! \sum_{k \geqslant 0} \frac{W_n(\nu; 2k)}{k!(k+\nu)!} \left(-\frac{x^2}{4}\right)^k,$$

which yields a fine alternative generating function for the even moments.

**Example 2.20.** In the case $k = 1$, Theorem 2.18 immediately implies that the second moment of an $n$-step random walk in any dimension is

$$W_n(\nu; 2) = n.$$

This was proved in [Wan13, Theorem 4.2] using a multi-dimensional integral representation and hyper-spherical coordinates. Similarly, we find that

$$W_n(\nu; 4) = \frac{n(n(\nu+2)-1)}{\nu+1}. \tag{28}$$

More generally, Theorem 2.18 shows that $W_n(\nu; 2k)$ is a polynomial of degree $k$ in $n$, with coefficients that are rational functions in $\nu$. For instance,

$$W_n(\nu; 6) = \frac{n(n^2(\nu+2)(\nu+3) - 3n(\nu+3) + 4)}{(\nu+1)^2} \tag{29}$$

and so on. ◇

Using the explicit expression of the even moments of an $n$-step random walk in dimension $d$, we derive the following convolution relation.

**Corollary 2.21. (Moment recursion)** *For positive integers $n_1, n_2$, half-integer $\nu$ and nonnegative integer $k$ we have*

$$W_{n_1+n_2}(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!} W_{n_1}(\nu; 2j) W_{n_2}(\nu; 2(k-j)). \tag{30}$$

Note the special case $n_2 = 1$, that is

$$W_n(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!} W_{n-1}(\nu; 2j), \tag{31}$$

which allows us to relate the moments of an $n$-step walk to the moments of an $(n-1)$-step walk.

**Example 2.22. (Integrality of two and four dimensional even moments)** Corollary 2.21 provides an efficient way to compute even moments of random walks in any dimension. For illustration, and because they are integral, we record the moments of an $n$-step walk in two and four dimensions for $n = 2, 3, \ldots, 6$.

$$\begin{aligned}
W_2(0; 2k) &: \quad 1, 2, 6, 20, 70, 252, 924, 3432, 12870, \ldots \\
W_3(0; 2k) &: \quad 1, 3, 15, 93, 639, 4653, 35169, 272835, 2157759, \ldots \\
W_4(0; 2k) &: \quad 1, 4, 28, 256, 2716, 31504, 387136, 4951552, 65218204, \ldots \\
W_5(0; 2k) &: \quad 1, 5, 45, 545, 7885, 127905, 2241225, 41467725, 798562125, \ldots \\
W_6(0; 2k) &: \quad 1, 6, 66, 996, 18306, 384156, 8848236, 218040696, 5651108226, \ldots
\end{aligned}$$

For $n = 2$, these are central binomial coefficients, see (36), while, for $n = 3, 4$, these are Apéry-like sequences, see (40) and (61). Likewise, the initial even moments for four dimensions are as follows.

$$\begin{aligned}
W_2(1; 2k) &: \quad 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots \\
W_3(1; 2k) &: \quad 1, 3, 12, 57, 303, 1743, 10629, 67791, 448023, \ldots \\
W_4(1; 2k) &: \quad 1, 4, 22, 148, 1144, 9784, 90346, 885868, 9115276, \ldots \\
W_5(1; 2k) &: \quad 1, 5, 35, 305, 3105, 35505, 444225, 5970725, 85068365, \ldots \\
W_6(1; 2k) &: \quad 1, 6, 51, 546, 6906, 99156, 1573011, 27045906, 496875786, \ldots
\end{aligned}$$

Observe that the first terms are as determined in Example 2.20. In the two-step case in four dimensions, we find that the even moments are the *Catalan numbers* $C_k$, that is

$$W_2(1; 2k) = \frac{(2k+2)!}{(k+1)!(k+2)!} = C_{k+1}, \quad C_k := \frac{1}{k+1}\binom{2k}{k}. \tag{32}$$

This adds another interpretation to the impressive array of quantities that are given by the Catalan numbers.

It is a special property of the random walks in two and four dimensions that all even moments are positive integers (compare, for instance, (28)). This is obvious for two dimensions from Theorem 2.18 which, in fact, demonstrates that the moments

$$W_n(0;2k) = \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1,\ldots,k_n}^2$$

count *abelian squares* [RS09]. On the other hand, to show that the four-dimensional moments $W_n(1;2k)$ are always integral, it suffices to recursively apply (31) and to note that the factors

$$\binom{k}{j}\frac{(k+1)!}{(k-j+1)!(j+1)!} = \frac{1}{j+1}\binom{k}{j}\binom{k+1}{j} \tag{33}$$

are integers for all nonnegative integers $j$ and $k$. The numbers (33) are known as *Narayana numbers* and occur in various counting problems; see, for instance, [Sta99, Problem 6.36]. ◇

**Example 2.23. (Narayana numbers)** The recursion (31) for the moments $W_n(\nu;2k)$ is equivalent to the following interpretation of the moments as row sums of the $n$th power of certain triangular matrices. Indeed, for given $\nu$, let $A(\nu)$ be the infinite lower triangular matrix with entries

$$A_{k,j}(\nu) = \binom{k}{j}\frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!} \tag{34}$$

for row indices $k=0,1,2,\ldots$ and column indices $j=0,1,2,\ldots$. Then the row sums of $A(\nu)^n$ are given by the moments $W_{n+1}(\nu;2k)$, $k=0,1,2,\ldots$. For instance, in the case $\nu=1$,

$$A(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \\ 1 & 3 & 1 & 0 & \\ 1 & 6 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}, \qquad A(1)^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \\ 12 & 9 & 1 & 0 & \\ 57 & 72 & 18 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

with the row sums $1,2,5,14,\ldots$ and $1,4,22,148,\ldots$ corresponding to the moments $W_2(1;2k)$ and $W_4(1;2k)$ as given in Example 2.22. Observe that, since the first column of $A(\nu)$ is composed of 1's, the sequence of moments $W_n(1;2k)$ can also be directly read off from the first column of $A(\nu)^n$. The matrix $A(1)$ is known as the *Narayana triangle* or the *Catalan triangle* [Slo15, `A001263`]. ◇

**Remark 2.24.** Let us note another point of view on the appearance of the Narayana triangle in the context of random walks. Let $\boldsymbol{X}$ be a random vector, which is uniformly distributed on the unit sphere in $\mathbb{R}^d$. If $\theta$ is the angle between $\boldsymbol{X}$ and a fixed axis, then $\Lambda = \cos\theta$ has the probability density [Kin63, (2)]

$$\frac{\nu!}{\sqrt{\pi}(\nu-1/2)!}(1-\lambda^2)^{\nu-1/2}, \quad \lambda \in [-1,1].$$

Denote with $R_n$ the random variable describing the distance to the origin after $n$ unit steps. Then $R_{n+1}$ is related to $R_n$ via [Kin63, (9)]

$$R_{n+1} = \sqrt{1+2\Lambda R_n + R_n^2}.$$

Writing $\mathbb{E}[X]$ for the expected value of a random variable $X$, we therefore have

$$W_{n+1}(\nu; 2k) = \mathbb{E}[R_{n+1}^{2k}] = \mathbb{E}[(1 + 2\Lambda R_n + R_n^2)^k].$$

In terms of the generalized *Narayana polynomials*

$$\mathcal{N}_k^{(\nu)}(z) = \mathbb{E}[(1 + 2\lambda\sqrt{z} + z)^{k-1}], \tag{35}$$

which were introduced in [AMV13, (6.2)] for $k \geqslant 1$, we obtain

$$W_{n+1}(\nu; 2k) = \mathbb{E}[\mathcal{N}_{k+1}^{(\nu)}(R_n^2)].$$

This recurrence identity on the moments can then be expressed in matrix form by defining a matrix $A(\nu)$ as in Example 2.23.

We note, moreover, that expressing the Narayana polynomials (35) in terms of the Gegenbauer polynomials $C_k^{\mu}(z)$, as demonstrated in [AMV13, Theorem 6.3], we deduce the expression of the even moments as

$$W_{n+1}(\nu; 2k) = \frac{k!}{(2\nu+1)_k}\mathbb{E}\left[(1 - R_n^2)^k C_k^{\nu+1/2}\left(\frac{1 + R_n^2}{1 - R_n^2}\right)\right],$$

which is a variation of (31).

**Example 2.25. (Six dimensional even moments)** To contrast with the integral even moments in 2 and 4 dimensions in Example 2.22, we record a few initial even moments in 6 dimensions.

$$\begin{aligned}
W_2(2; 2k) &\quad : \quad 1, 2, 14/3, 12, 33, 286/3, 286, 884, 8398/3, \ldots \\
W_3(2; 2k) &\quad : \quad 1, 3, 11, 139/3, 216, 1088, 5825, 32763, 191935, \ldots \\
W_4(2; 2k) &\quad : \quad 1, 4, 20, 352/3, 2330/3, 16952/3, 133084/3, 370752, 3265208, \ldots
\end{aligned}$$

It may be concluded from (34), with $\nu = 2$, that the entries of the matrix $A(2)$ satisfy $A_{k,j}(2) \in \frac{1}{3}\mathbb{Z}$. This implies that the even moments in dimension 6 are rational numbers whose denominators are powers of 3. Similar observations apply in all dimensions but we do not pursue this theme further here. $\diamond$

# 3 Moments of short walks

## 3.1 Moments of 2-step walks

It follows from Theorem 2.18 that the general expression of the even moments for a 2-step walk in $d$ dimensions is given by

$$W_2(\nu; 2k) = \frac{\binom{2k+2\nu}{k}}{\binom{k+\nu}{k}} = \frac{\nu!(2k+2\nu)!}{(k+\nu)!(k+2\nu)!}. \tag{36}$$

**Example 3.1.** Note that in the special case of $\nu = 0$, that is, dimension 2, this clearly reduces to the central binomial coefficient. In dimension 4, as noted in Example 2.22, the even moments are the Catalan numbers. We note that the generating function for the two-step even moments is

$$\sum_{k=0}^{\infty} W_2(\nu; 2k)x^k = {}_2F_1\left(\left.\begin{matrix}1, \nu + \frac{1}{2} \\ 2\nu + 1\end{matrix}\right| 4x\right), \tag{37}$$

which reduces to the known generating functions for $\nu = 0, 1$. Equation (37) is an immediate consequence of rewriting (36) as $W_2(\nu; 2k) = 2^{2k} \frac{(\nu+1/2)_k}{(2\nu+1)_k}$. $\diamond$

In fact, (36) also holds true when $k$ takes complex values. This was proved in [Wan13, Theorem 4.3] using a multi-dimensional integral representation and hyper-spherical coordinates. We offer an alternative proof based on Theorem 2.14.

**Theorem 3.2.** *For all complex $s$ and half-integer $\nu \geqslant 0$,*

$$W_2(\nu; s) = \frac{\nu!\Gamma(s + 2\nu + 1)}{\Gamma\left(\frac{s}{2} + \nu + 1\right)\Gamma\left(\frac{s}{2} + 2\nu + 1\right)}.$$

*Proof.* The case $k = 0$ of (25) in Theorem 2.14 gives

$$W_n(\nu; s) = \frac{2^{s+1}\Gamma\left(\frac{s}{2} + \nu + 1\right)}{\Gamma(\nu + 1)\Gamma\left(-\frac{s}{2}\right)} \int_0^\infty x^{-s-1} j_\nu^n(x)\mathrm{d}x,$$

provided that $-n(\nu + 1/2) - 1 < s < 0$. Using that, for $0 < \mathrm{Re}\ s < 2\nu + 1$,

$$\int_0^\infty x^{s-1} j_\nu^2(x)\mathrm{d}x = 2^{2\nu-1} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2} + \nu\right)\Gamma(1 + \nu)^2}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1 - \frac{s}{2} + \nu\right)\Gamma\left(1 - \frac{s}{2} + 2\nu\right)}, \tag{38}$$

the claimed formula then follows from the duplication formula for the gamma function and analytic continuation. $\square$

## 3.2 Moments of 3-step walks

**Lemma 3.3.** *The nonnegative even moments for a 3-step walk in $d$ dimensions are*

$$W_3(\nu; 2k) = \sum_{j=0}^{k} \binom{k}{j}\binom{k+\nu}{j}\binom{2j+2\nu}{j}\binom{j+\nu}{j}^{-2}. \tag{39}$$

*Proof.* We apply Corollary 2.21 with $n_1 = 2$ and $n_2 = 1$. Using that $W_1(\nu; 2k) = 1$ and that an evaluation of $W_2(\nu; 2k)$ is given by (36), we obtain

$$\begin{aligned} W_3(\nu; 2k) &= \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!} W_2(\nu; 2j) \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!} \frac{\nu!(2j+2\nu)!}{(j+\nu)!(j+2\nu)!}. \end{aligned}$$

Expressing the factorials as binomial coefficients yields (39). $\square$

**Example 3.4. (Generating function for 3 steps in 2 dimensions)** In the case $d = 2$, or $\nu = 0$, the moments of a 3-step walk reduce to the Apéry-like numbers

$$W_3(0; 2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}. \tag{40}$$

[BSWZ12, (3.2) & (3.4)] show that the generating function for this sequence is

$$\sum_{k=0}^{\infty} W_3(0;2k)x^k = \frac{1}{1+3x}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\right|\frac{27x(1-x)^2}{(1+3x)^3}\right).$$

◇

**Example 3.5. (Generating function for 3 steps in 4 dimensions)** In the case $d=4$, or $\nu=1$, the moments, whose initial values are recorded in Example 2.22, are sequence [Slo15, `A103370`]. The OEIS also records a hypergeometric form of the generating function (as the linear combination of a hypergeometric function and its derivative), added by Mark van Hoeij. On using linear transformations of hypergeometric functions, we have more simply that

$$\sum_{k=0}^{\infty} W_3(1;2k)x^k = \frac{1}{2x^2} - \frac{1}{x} - \frac{(1-x)^2}{2x^2(1+3x)}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 2\end{matrix}\right|\frac{27x(1-x)^2}{(1+3x)^3}\right),$$

which we are able to generalize. ◇

Example 3.5 suggests that a nice formula for the generating function for the moments $W_3(\nu;2k)$ exists for all even dimensions. Indeed, we have the following result.

**Theorem 3.6. (Ordinary generating function for even moments with three steps)** *For integers $\nu \geqslant 0$ and $|x| < 1/9$, we have*

$$\sum_{k=0}^{\infty} W_3(\nu;2k)x^k = \frac{(-1)^\nu}{\binom{2\nu}{\nu}} \frac{(1-1/x)^{2\nu}}{1+3x}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1+\nu\end{matrix}\right|\frac{27x(1-x)^2}{(1+3x)^3}\right) - q_\nu\left(\frac{1}{x}\right), \tag{41}$$

*where $q_\nu(x)$ is a polynomial (that is, $q_\nu(1/x)$ is the principal part of the hypergeometric term on the right-hand side).*

*Proof.* For integers $\nu \geqslant 0$, define the rational numbers $H(\nu;k)$ by

$$\frac{(-1)^\nu}{\binom{2\nu}{\nu}} \frac{(1-1/x)^{2\nu}}{1+3x}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1+\nu\end{matrix}\right|\frac{27x(1-x)^2}{(1+3x)^3}\right) = \sum_{k=-2\nu}^{\infty} H(\nu;k)x^k \tag{42}$$

for $k \geqslant -2\nu$, and $H(\nu;k)=0$ for $k<-2\nu$. Writing the sum (39) for $W_3(\nu;2k)$ in hypergeometric form, we obtain, for integers $k,\nu \geqslant 0$, the representation

$$W_3(\nu;2k) = {}_3F_2\left(\left.\begin{matrix}-k,-k-\nu,\nu+1/2\\ \nu+1,2\nu+1\end{matrix}\right|4\right). \tag{43}$$

In order to prove the claimed generating function (41) it therefore suffices to show the (more precise) claim

$$H(\nu;k) = \operatorname{Re}\,{}_3F_2\left(\left.\begin{matrix}-k,-k-\nu,\nu+1/2\\ \nu+1,2\nu+1\end{matrix}\right|4\right). \tag{44}$$

For instance, this predicts that $q_2(x) = 1/6 - 5/6\,x + x^2 + 1/3\,x^3 + x^4$.

Taking the real part is only necessary for $k<-\nu$ while, for $k \geqslant -\nu$, the ${}_3F_2$ is terminating. We note, as will be demonstrated later in the proof, that the right-hand side of (44) vanishes for $k<-2\nu$, that is, the ${}_3F_2$ takes purely imaginary values then.

The holonomic systems approach [Zei90], implemented in the *Mathematica* package `HolonomicFunctions`, which accompanies Koutschan's thesis [Kou09], shows that the coefficients $H(\nu;k)$, defined by (42), satisfy the recursive relation

$$9(k+1)(k+\nu+1)H(\nu;k) = \tfrac{1}{2}\left(20\left(k+\tfrac{3}{2}\right)^2 + 60\left(k+\tfrac{3}{2}\right)\nu + 36\nu^2 + 1\right)H(\nu;k+1) - (k+2\nu+2)(k+3\nu+2)H(\nu;k+2) \tag{45}$$

for all integers $k$ and all integers $\nu \geqslant 0$. We already know that (44) holds for $\nu = 0$ and $k \geqslant 0$. Verifying, by using

$${}_3F_2\left(\left.\begin{matrix}1,1,1/2\\1,1\end{matrix}\right|4x\right) = \frac{1}{\sqrt{1-4x}}$$

and letting $x \to 1$ to see that the real part vanishes (for any choice of analytic continuation to $x = 1$), that (44) holds for $\nu = 0$ and $k = -1$, we conclude from (45) that (44) is indeed true for $\nu = 0$ and all integers $k$.

As in the case of (45), we find that the coefficients $H(\nu;k)$ further satisfy the dimensional relations

$$\begin{aligned}&2(k+2\nu)(k+3\nu-1)(k+3\nu)H(\nu;k)\\ = \;&\nu^2(7k+15\nu-4)H(\nu-1;k+1) - 9\nu^2(k+\nu)H(\nu-1;k)\end{aligned} \tag{46}$$

and

$$2(k+1)(k+3\nu)H(\nu;k) = \nu^2 H(\nu-1;k+2) - 3\nu^2 H(\nu-1;k+1) \tag{47}$$

for all integers $k$ and all integers $\nu \geqslant 1$. The three relations (45), (46), (47), together with the case $\nu = 0$ as boundary values, completely determine the coefficients $H(\nu;k)$ for all integers $k$ and $\nu \geqslant 1$.

Since we already verified the case $\nu = 0$, it only remains to demonstrate that the right-hand side of (44) satisfies the same recursive relations. Another application of `HolonomicFunctions` finds that the ${}_3F_2$ on right-hand side of (44), and hence its real part, indeed satisfy (45), (46), (47) for the required (real) values of $\nu$ and $k$. □

For the convenience of the reader, and because we will frequently use it in the following, we state Carlson's Theorem next [Tit39, 5.81]. Recall that a function $f(z)$ is of *exponential type* in a region if $|f(z)| \leqslant Me^{c|z|}$ for some constants $M$ and $c$.

**Theorem 3.7. (Carlson's Theorem)** *Let $f$ be analytic in the right half-plane* $\operatorname{Re} z \geqslant 0$ *and of exponential type with the additional requirement that*

$$|f(z)| \leqslant Me^{d|z|}$$

*for some $d < \pi$ on the imaginary axis* $\operatorname{Re} z = 0$. *If $f(k) = 0$ for $k = 0, 1, 2, \ldots$, then $f(z) = 0$ identically.*

**Example 3.8.** Applying creative telescoping to the binomial sum (39), we derive that the moments $W_3(\nu;2k)$ satisfy the recursion

$$\begin{aligned}&(k+2\nu+1)(k+3\nu+1)W_3(\nu;2k+2)\\ = \;&\tfrac{1}{2}\left(20\left(k+\tfrac{1}{2}\right)^2 + 60\left(k+\tfrac{1}{2}\right)\nu + 36\nu^2 + 1\right)W_3(\nu;2k) - 9k(k+\nu)W_3(\nu;2k-2).\end{aligned} \tag{48}$$

Observe that $W_n(\nu;s)$ is analytic for $\operatorname{Re} s \geqslant 0$ and is bounded in that half-plane by $|W_n(\nu;s)| \leqslant n^{\operatorname{Re} s}$ (because, in any dimension, the distance after $n$ random steps is bounded by $n$). It therefore follows from Carlson's Theorem, as detailed in [BNSW11, Theorem 4], that the recursion (48) remains valid for complex values of $k$. $\diamond$

The next result expresses the complex moments $W_3(\nu;s)$ in terms of a Meijer $G$-function and extends [BSW13, Theorem 2.7].

**Theorem 3.9. (Meijer $G$ form of $W_3$)** *For all complex $s$ and dimensions $d \geqslant 2$,*

$$W_3(\nu;s) = 2^{2\nu}\nu!^2 \frac{\Gamma\left(\frac{s}{2}+\nu+1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{s}{2}\right)} G^{3,3}_{2,1}\left(\begin{matrix}1, 1+\nu, 1+2\nu \\ \frac{1}{2}+\nu, -\frac{s}{2}, -\frac{s}{2}-\nu\end{matrix}\middle|\frac{1}{4}\right).$$

*Proof.* If $0 < \operatorname{Re}(s) < \nu + \frac{3}{2}$, then

$$\int_0^\infty x^{s-1} j_\nu(x) \mathrm{d}x = 2^{s-1} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma(1+\nu)}{\Gamma\left(1-\frac{s}{2}+\nu\right)}.$$

This simple integral is a consequence of the fact that $W_1(\nu;s) = 1$ combined with Theorem 2.14 with $n = 1$ and $k = 0$. Similarly, for $0 < \operatorname{Re}(s) < 2\nu + 1$, the Mellin transform of $j_\nu^2(x)$ is given by (38). Applying Parseval's formula to these two Mellin transforms, we obtain, for $0 < \delta < 1$,

$$\begin{aligned}
&\int_0^\infty x^{s-1} j_\nu^3(x) \mathrm{d}x \\
=\ & \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} 2^{2\nu-1} \frac{\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{1-z}{2}+\nu\right)\Gamma(1+\nu)^2}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{z}{2}+\nu\right)\Gamma\left(1-\frac{z}{2}+2\nu\right)} 2^{s-z-1} \frac{\Gamma\left(\frac{s-z}{2}\right)\Gamma(1+\nu)}{\Gamma\left(1-\frac{s-z}{2}+\nu\right)} \mathrm{d}z \\
=\ & \frac{2^{2\nu+s-1}\Gamma(1+\nu)^3}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{2\pi i}\int_{\delta/2-i\infty}^{\delta/2+i\infty} \frac{2^{-2t}\Gamma(t)\Gamma\left(\frac{1}{2}-t+\nu\right)}{\Gamma(1-t+\nu)\Gamma(1-t+2\nu)} \frac{\Gamma\left(\frac{s}{2}-t\right)}{\Gamma\left(1-\frac{s}{2}+t+\nu\right)} \mathrm{d}t \\
=\ & \frac{2^{2\nu+s-1}\nu!^3}{\Gamma\left(\frac{1}{2}\right)} G^{3,3}_{2,1}\left(\begin{matrix}1, 1+\nu, 1+2\nu \\ \frac{1}{2}+\nu, \frac{s}{2}, \frac{s}{2}-\nu\end{matrix}\middle|\frac{1}{4}\right).
\end{aligned}$$

The claim then follows, by analytic continuation, from Theorem 2.14 with $n = 3$ and $k = 0$. $\square$

We note that, as in [BSW13], this Meijer $G$-function expression can be expressed as a sum of hypergeometric functions by Slater's Theorem [Mar83, p. 57]. This is made explicit in (51).

Equation (43) gives a hypergeometric expression for the even moments of a 3-step random walk. It was noticed in [BNSW11] that, in the case of planar walks, the *real part* of this hypergeometric expression provides an evaluation of the odd moments. These odd moments are much harder to obtain, and it was first proved in [BNSW11], based on this observation, that the average distance of a planar 3-step random walk is

$$W_3(0;1) = A + \frac{6}{\pi^2}\frac{1}{A} \approx 1.5746, \tag{49}$$

where

$$W_3(0;-1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\tfrac{1}{3}\right) =: A. \tag{50}$$

In the sequel, we generalize these results from two to arbitrary even dimensions. In particular, as explained in Example 3.12, we establish the transcendental nature of the odd moments of 3-step walks in all even dimensions by showing that they are all rational linear combinations of $A$ and $1/(\pi^2 A)$.

**Theorem 3.10. (Hypergeometric form of $W_3$ at odd integers)** *Suppose that $d$ is even, that is, $\nu$ is an integer. For all odd integers $s \geqslant -2\nu - 1$,*

$$W_3(\nu; s) = \mathrm{Re}\ {}_3F_2\left(\left.\begin{matrix} -s/2, -s/2-\nu, \nu+1/2 \\ \nu+1, 2\nu+1 \end{matrix}\right| 4\right).$$

*Proof.* The case $\nu = 0$ is proved in [BNSW11, Theorem 6]. We will prove the general case by induction on $\nu$.

It is routine to verify that the hypergeometric function

$$F(\nu; s) = {}_3F_2\left(\left.\begin{matrix} -s/2, -s/2-\nu, \nu+1/2 \\ \nu+1, 2\nu+1 \end{matrix}\right| 4\right),$$

which, for even $s$, agrees with $W_3(\nu; s)$ by (43), satisfies the contiguity relation

$$(s+1)(s+6\nu-1)F(\nu; s-1) + 6\nu^2 F(\nu-1; s+1) - 2\nu^2 F(\nu-1; s+3) = 0.$$

On the other hand, it follows from Theorem 3.9 and Slater's Theorem [Mar83, p. 57] that

$$\begin{aligned} W_3(\nu; s) \quad = \quad & \frac{\Gamma\left(-\nu - \frac{s+1}{2}\right)\Gamma(\nu+1)^2\Gamma\left(\nu + \frac{s}{2} + 1\right)}{2\pi\Gamma\left(-\frac{s}{2}\right)\Gamma\left(2\nu + \frac{s+3}{2}\right)} {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, \frac{1}{2}-\nu, \frac{1}{2}+\nu \\ \nu+\frac{s+3}{2}, 2\nu+\frac{s+3}{2} \end{matrix}\right| \frac{1}{4}\right) \\ & + \frac{2^{s+2\nu}\Gamma(\nu+1)\Gamma\left(\nu+\frac{s+1}{2}\right)}{\sqrt{\pi}\Gamma\left(2\nu+\frac{s}{2}+1\right)} {}_3F_2\left(\left.\begin{matrix} -\frac{s}{2}, -\frac{s}{2}-\nu, -\frac{s}{2}-2\nu \\ \nu+1, -\nu-\frac{s-1}{2} \end{matrix}\right| \frac{1}{4}\right). \end{aligned} \tag{51}$$

In that form, it is again a routine application of the holonomic systems approach [Zei90] to derive that

$$\begin{aligned} 0 \quad = \quad & (s+1)(s+6\nu-1)W_3(\nu; s-1) \\ & + 6\nu^2 W_3(\nu-1; s+1) - 2\nu^2 W_3(\nu-1; s+3). \end{aligned} \tag{52}$$

Since this relation matches the relation satisfied by $F(\nu; s)$, and hence Re $F(\nu; s)$ when $\nu$ and $s$ are real, the general case follows inductively from the base case $\nu = 0$. $\square$

**Remark 3.11.** The coefficients of the hypergeometric functions in (51) can be expressed as

$$c_1 := \frac{\tau(\nu; s)}{\pi} \frac{2^{2\nu-1}}{(2\nu+1)\binom{2\nu}{\nu}} \frac{\binom{2\nu+s+1}{\nu+(s+1)/2}}{\binom{s+2\nu}{s/2}\binom{4\nu+s+1}{2\nu+1}}$$

and $c_2 := \binom{2\nu+s}{\nu+s/2}/\binom{2\nu+s/2}{\nu}$, respectively. Here, the factor

$$\tau(\nu; s) = \frac{1}{\cos(\pi\nu)\cot(\pi s/2) - \sin(\pi\nu)}$$

is $\pm 1$ for half-integers $\nu$, and $\pm\tan(\frac{\pi s}{2})$ for integers $\nu$.

**Example 3.12. (Odd moments $W_3(\nu;\cdot)$ in even dimensions)** The planar case of Theorem 3.10 was used in [BNSW11] to prove that the average distance to the origin after three random steps in the plane is given by (49). It is a consequence of (48), extended to complex $k$, that all planar odd moments are $\mathbb{Q}$-linear combinations of $A = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6(1/3)$, defined in (50), and $1/(\pi^2 A)$.

The dimensional recursion (52) used in the proof of Theorem 3.10 shows that this observation extends to all even dimensions. For instance,

$$W_3(1;-3) = \frac{4}{3}A - \frac{4}{\pi^2}\frac{1}{A}, \quad W_3(1;-1) = \frac{4}{15}A + \frac{4}{\pi^2}\frac{1}{A}.$$

Moreover, the average distance to the origin after three random steps in four dimensions is

$$W_3(1;1) = \frac{476}{525}A + \frac{52}{7\pi^2}\frac{1}{A} \approx 1.6524,$$

with similar evaluations in six or higher even dimensions. ◇

**Example 3.13.** Theorem 3.10 does not hold in odd dimensions, in which the involved quantities can be evaluated in elementary terms. For instance, in the case of dimension 3,

$$W_3(\tfrac{1}{2};s) = \frac{1}{4}\frac{3^{s+3}-3}{(s+2)(s+3)}$$

while for integer $s>0$

$${}_3F_2\left(\left.\begin{matrix}-s/2, -s/2-1/2, 1\\ 3/2, 2\end{matrix}\right| 4\right) = \frac{1}{4}\frac{3^{s+3}-2-(-1)^s}{(s+2)(s+3)}$$

only agrees for even $s$. ◇

**Example 3.14. (First derivative of $W_3(\nu;\cdot)$ in even dimensions)** By differentiating the hypergeometric representation of $W_3(\nu;s)$ in (51), it was shown in [BSWZ12, Examples 6.2 and 6.6] that

$$W_3'(0;0) = \frac{1}{\pi}\operatorname{Cl}\left(\frac{\pi}{3}\right), \quad W_3'(0;2) = 2 + \frac{3}{\pi}\operatorname{Cl}\left(\frac{\pi}{3}\right) - \frac{3\sqrt{3}}{2\pi},$$

where the derivatives are with respect to $s$. It follows from differentiating (48), extended to complex $k$, that all derivatives $W_3'(0;2k)$ lie in the $\mathbb{Q}$-linear span of 1, $\frac{1}{\pi}\operatorname{Cl}(\frac{\pi}{3})$ and $\frac{\sqrt{3}}{\pi}$. Then, differentiating (52), we find that, indeed, for all integers $\nu \geqslant 0$, the derivatives $W_3'(\nu;2k)$ can likewise be expressed as

$$W_3'(\nu;2k) = r_1 + r_2\frac{\sqrt{3}}{\pi} + r_3\frac{1}{\pi}\operatorname{Cl}(\frac{\pi}{3}),$$

with rational numbers $r_1, r_2, r_3$.

Moreover, in the case of $W_3'(\nu;0)$, a slightly more careful analysis reveals that $r_3 = 1$. While we omit the details, we note that this can be seen, for instance, by evaluating $W_3'(1;0)$ and then deriving, in analogy with (52), a functional equation relating $W_3(\nu;s)$, $W_3(\nu+1;s)$ and $W_3(\nu+2;s)$. In four and six dimensions, we obtain, for example,

$$W_3'(1;0) = \frac{1}{2} - \frac{11\sqrt{3}}{16\pi} + \frac{1}{\pi}\operatorname{Cl}\left(\frac{\pi}{3}\right), \quad W_3'(2;0) = \frac{17}{36} - \frac{181\sqrt{3}}{320\pi} + \frac{1}{\pi}\operatorname{Cl}\left(\frac{\pi}{3}\right).$$

A special motivation for considering these derivative values is that, in the case of two dimensions, $W_3'(0;0)$ is the (logarithmic) Mahler measure of the multivariate polynomial $1+x_1+x_2$; see [BSWZ12, Example 6.6] or [BS12, Section 4]. ◇

**Example 3.15. (Second derivative of $W_3(\nu;\cdot)$ in even dimensions)** The second derivative $W_3''(0;0)$, interpreted there as a higher Mahler measure, is evaluated in [BS12, Theorem 4.4] in the form

$$W_3''(0;0) = \frac{\pi^2}{4} + \frac{3}{\pi}\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right), \tag{53}$$

where $\mathrm{Ls}_n$ denotes the $n$th *log-sine integral*

$$\mathrm{Ls}_n(\sigma) := -\int_0^\sigma \log^{n-1}\left|2\sin\frac{\theta}{2}\right|\mathrm{d}\theta.$$

For alternative expressions of the log-sine integral in (53) in terms of other polylogarithmic constants, we refer to [BS12]. We have not been able to obtain an equally natural log-sine evaluation of

$$W_3''(0;2k) = \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi} |1+e^{i\omega}+e^{i\theta}|^{2k}\log^2|1+e^{i\omega}+e^{i\theta}|\,\mathrm{d}\theta\mathrm{d}\omega, \tag{54}$$

for integers $k>0$. We may, however, derive

$$W_3''(0;2) = 3\,W_3''(0;0) + \frac{3}{\pi^2}\int_0^{\pi}\cos\omega\ \mathrm{Re}\int_0^{2\pi}\log^2\left(1-2\sin\left(\frac{\omega}{2}\right)e^{i\theta}\right)\,\mathrm{d}\theta\mathrm{d}\omega. \tag{55}$$

By the methods of [BS12, §4], we arrive at

$$W_3''(0;2) = 3\,W_3''(0;0) + \frac{\sqrt{3}}{2}\,\pi - \frac{3}{2} - \frac{6}{\pi}\int_0^{\pi/3}\mathrm{Li}_2\left(4\,\sin^2\left(\frac{\omega}{2}\right)\right)\cos\omega\,\mathrm{d}\omega. \tag{56}$$

We may now integrate by parts and obtain

$$W_3''(0;2) = 3\,W_3''(0;0) - 1 - \frac{2\sqrt{3}}{\pi}\int_0^1\sqrt{\frac{1+\frac{s}{3}}{1-s}}\,\log s\,\mathrm{d}s. \tag{57}$$

Moreover,

$$\int_0^1\sqrt{\frac{1+\frac{s}{3}}{1-s}}\,\log s\,\mathrm{d}s = \sum_{n=0}^\infty \frac{a_n}{3^n\,n^2} \tag{58}$$

where $a_n$ is given by [Slo15, `A025565`] of the OEIS and counts "the number of number of UDU-free paths of $n-1$ upsteps (U) and $n-1$ downsteps (D)" with recursion

$$(n-1)a_n - 2(n-1)a_{n-1} - 3(n-3)a_{n-2} = 0.$$

(Alternatively, $a_n = M_{n-1} + \sum_{k=1}^{n-1} M_{k-1}a_{n-k}$ with $M_k$ the *Motzkin numbers* given in `A001006`.) Solving for the generating function $a_3(x)$ of $a_n$ and considering $\int_0^{1/2}\int_0^t a_3(x)\mathrm{d}x\frac{\mathrm{d}t}{t}$ we finally arrive at

$$W_3''(0;2) = 3\,W_3''(0;0) - 3\,\frac{\sqrt{3}}{\pi}(\log 3 - 1) - \frac{1}{2} + \frac{4}{\pi}\mathrm{Cl}\left(\frac{\pi}{3}\right). \tag{59}$$

As in Example 3.14, all second derivatives $W_3''(\nu;2k)$ in even dimensions may then be expressed in terms of $W_3''(0;0)$ and $W_3''(0;2)$ as well as the constants in Example 3.14. For instance,

$$W_3''(1;0) = -\frac{3}{8}W_3''(0;0) + \frac{11}{24}W_3''(0;2) - \frac{3}{8} + \frac{23\sqrt{3}}{48\pi} - \frac{5}{6\pi}\,\mathrm{Cl}\left(\frac{\pi}{3}\right),$$

and so on. $\diamond$

### 3.3 Moments of 4-step walks

**Lemma 3.16.** *The nonnegative even moments for a 4-step walk in $d$ dimensions are*

$$W_4(\nu;2k)=\sum_{j=0}^{k}\frac{\binom{k}{j}\binom{k+\nu}{j}}{\binom{j+\nu}{j}}\frac{\binom{2j+2\nu}{j}}{\binom{j+\nu}{j}}\frac{\binom{2(k-j)+2\nu}{k-j}}{\binom{k-j+\nu}{k-j}}. \tag{60}$$

*Proof.* We apply Corollary 2.21 with $n_1=2$ and $n_2=2$, to obtain

$$W_4(\nu;2k)=\sum_{j=0}^{k}\frac{\binom{k}{j}\binom{k+\nu}{j}}{\binom{j+\nu}{j}}W_2(\nu;2j)W_2(\nu;2(k-j)).$$

Using the evaluation of $W_2(\nu;2k)$ given by (36) then yields (60). □

**Example 3.17. (Generalised Domb numbers)** The binomial sums in (60) generalize the *Domb numbers*, also known as the *diamond lattice* numbers [Slo15, `A002895`],

$$W_4(0;2k)=\sum_{j=0}^{k}\binom{k}{j}^2\binom{2j}{j}\binom{2(k-j)}{k-j}, \tag{61}$$

for $k=0,1,2,\ldots$, which have played an important role in dimension 2. Their ordinary generating function,

$$\begin{aligned}\sum_{k=0}^{\infty}W_4(0;2k)x^k&=\frac{1}{1-16x}\,{}_3F_2\left(\left.\begin{matrix}\frac13,\frac12,\frac23\\ 1,1\end{matrix}\right|\frac{108x}{(16x-1)^3}\right)\\ &=\frac{1}{1-16x}\,{}_2F_1\left(\left.\begin{matrix}\frac16,\frac13\\ 1\end{matrix}\right|\frac{108x}{(16x-1)^3}\right)^2,\end{aligned} \tag{62}$$

was determined in [Rog09]. The final equation follows from Clausen's product formula.

In four dimensions, the recursive relation (30) combined with (32) yields

$$W_4(1;2k)=\sum_{j=1}^{k+1}N(k+1,j)\,C_jC_{k-j+2},$$

where $C_k$ are the Catalan numbers, as in (32), and

$$N(k+1,j+1)=\frac{1}{j+1}\binom{k}{j}\binom{k+1}{j}$$

are the Narayana numbers, as in Example 3.17. After developing some further properties of the moments, we illustrate in Example 3.25 that the ordinary generating function for the even moments $W_4(\nu;2k)$ can be expressed in terms of hypergeometric functions whenever the dimension is even. ◇

**Corollary 3.18. (Hypergeometric form of $W_4$ at even integers)** *For $k=0,1,2,\ldots$, we have*

$$W_4(\nu;2k)=\frac{\binom{2k+2\nu}{k}}{\binom{k+\nu}{k}}\,{}_4F_3\left(\left.\begin{matrix}-k,-k-\nu,-k-2\nu,\nu+1/2\\ \nu+1,2\nu+1,-k-\nu+1/2\end{matrix}\right|1\right).$$

We note that this hypergeometric function is *well-poised* [DLMF, §16.4].

**Example 3.19.** Applying creative telescoping to the binomial sum (60), we derive that the moments $W_4(\nu;2k)$ satisfy the recursion

$$\begin{aligned}
& (k+2\nu+1)(k+3\nu+1)(k+4\nu+1)W_4(\nu;2k+2) \\
= \; & \left(\left(k+\tfrac{1}{2}\right)+2\nu\right)\left(20\left(k+\tfrac{1}{2}\right)^2+80\left(k+\tfrac{1}{2}\right)\nu+48\nu^2+3\right)W_4(\nu;2k) \\
& -64k(k+\nu)(k+2\nu)W_4(\nu;2k-2). \qquad (63)
\end{aligned}$$

As in Example 3.8, $W_4(\nu;s)$ is analytic, exponentially bounded for $\operatorname{Re} s \geqslant 0$ and bounded on vertical lines. Hence, Carlson's Theorem 3.7 applies to show that the recursion (63) extends to complex $k$. $\diamond$

The following result is the counterpart of Theorem 3.9 and extends [BSW13, Theorem 2.8].

**Theorem 3.20. (Meijer $G$ form of $W_4$)** *For all complex $s$ with $\operatorname{Re}(s) > -4\nu-2$ and dimensions $d \geqslant 2$,*

$$W_4(\nu;s) = 2^{s+4\nu}\nu!^3\frac{\Gamma\left(\frac{s}{2}+\nu+1\right)}{\Gamma\left(\frac{1}{2}\right)^2\Gamma\left(-\frac{s}{2}\right)}G^{4,4}_{2,2}\left(\begin{matrix}1,\frac{1-s}{2}-\nu,1+\nu,1+2\nu\\ \frac{1}{2}+\nu,-\frac{s}{2},-\frac{s}{2}-\nu,-\frac{s}{2}-2\nu\end{matrix}\middle|1\right).$$

*Proof.* The proof is obtained along the lines of the proof of Theorem 3.9. This time, Parseval's formula is applied to the product $j_\nu^4(x) = j_\nu^2(x)j_\nu^2(x)$. $\square$

**Remark 3.21.** We note that, as in the case of (51) for three steps, the Meijer $G$-function in Theorem 3.20 can be expressed as a sum of hypergeometric functions, namely

$$\begin{aligned}
W_4(\nu;s) \;=\; & d_1\cdot {}_4F_3\left(\begin{matrix}\frac{1}{2},\frac{1}{2}-\nu,\frac{1}{2}+\nu,2\nu+\frac{s}{2}+1\\ \nu+\frac{s+3}{2},2\nu+\frac{s+3}{2},3\nu+\frac{s+3}{2}\end{matrix}\middle|1\right) \\
& +d_2\cdot {}_4F_3\left(\begin{matrix}-\frac{s}{2},-\frac{s}{2}-\nu,-\frac{s}{2}-2\nu,\nu+\frac{1}{2}\\ \nu+1,\nu+2,-\nu-\frac{s-1}{2}\end{matrix}\middle|1\right), \qquad (64)
\end{aligned}$$

by Slater's Theorem [Mar83, p. 57]. The coefficients $d_1$ and $d_2$ of the two hypergeometric functions are

$$d_1 := \frac{2^{4\nu+s+1}\Gamma(\nu+1)\Gamma(2\nu+\frac{s}{2}+1)}{\sqrt{\pi}\Gamma(3\nu+\frac{s+3}{2})}c_1, \quad d_2 := c_2,$$

where the coefficients $c_1$ and $c_2$ are as in the three-step case, see Remark 3.11.

**Example 3.22. (Odd moments $W_4(\nu;\cdot)$ in even dimensions)** It was shown in [BSW13, Section 2.3 & 3.1] that the average distance to the origin after four random steps in the plane, as well as all its odd moments, can be evaluated in terms of the elliptic integrals

$$\begin{aligned}
A \;&:=\; \frac{1}{\pi^3}\int_0^1 K'(k)^2\mathrm{d}k = \frac{\pi}{16}{}_7F_6\left(\begin{matrix}\frac{5}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{1}{4},1,1,1,1,1\end{matrix}\middle|1\right), \\
B \;&:=\; \frac{1}{\pi^3}\int_0^1 k^2K'(k)^2\mathrm{d}k = \frac{3\pi}{256}{}_7F_6\left(\begin{matrix}\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},2,2,2,2,1\end{matrix}\middle|1\right).
\end{aligned}$$

We note that each of the ${}_7F_6$ hypergeometric functions may alternatively be expressed as the sum of two ${}_6F_5$ hypergeometric functions. Then, in two dimensions,

$$W_4(0;-1) = 4A, \quad W_4(0;1) = 16A - 48B,$$

and it follows from (63), extended to complex values, that all odd moments are indeed rational linear combinations of $A$ and $B$. In order to generalize this observation to higher dimensions, we claim that

$$\begin{aligned} 0 \;=\;& 3(s+2)(s+6\nu)(s+8\nu)(s+8\nu-2)W_4(\nu;s) \\ &+256\nu^3(s+4\nu)W_4(\nu-1;s+2) - 8\nu^3(5s+32\nu-6)W_4(\nu-1;s+4). \end{aligned} \tag{65}$$

This counterpart of (52) can be proved using the holonomic systems approach [Zei90] applied to the hypergeometric form (64). We conclude that, in any even dimension, the odd moments lie in the $\mathbb{Q}$-span of the constants $A$ and $B$, which arose in the planar case. For instance, we find that the average distance after four random steps in four dimensions is

$$W_4(1;1) = \frac{3334144}{165375}A - \frac{11608064}{165375}B,$$

and so on. $\diamond$

**Example 3.23. (First derivative of $W_4(\nu;\cdot)$ in even dimensions)** In continuation of Example 3.14, we recall from [BSWZ12, Examples 6.2 and 6.6] that

$$W_4'(0;0) = \frac{7\zeta(3)}{2\pi^2}, \quad W_4'(0;2) = \frac{14\zeta(3)}{\pi^2} - \frac{12}{\pi^2} + 3,$$

where, again, the derivatives are with respect to $s$. These evaluations may, for instance, be obtained from differentiating the hypergeometric expression (64). Proceeding as in Example 3.14, we differentiate both (63), extended to complex values, and (65), we conclude that, for all integers $k, \nu \geqslant 0$,

$$W_4'(\nu;2k) = r_1 + r_2\frac{1}{\pi^2} + r_3\frac{7\zeta(3)}{2\pi^2},$$

with rational numbers $r_1, r_2, r_3$. Again, we find that $r_3 = 1$ in the case $k = 0$. In four and six dimensions, we obtain, for example,

$$W_4'(1;0) = \frac{3}{4} - \frac{53}{9\pi^2} + \frac{7\zeta(3)}{2\pi^2}, \quad W_4'(2;0) = \frac{13}{24} - \frac{48467}{14175\pi^2} + \frac{7\zeta(3)}{2\pi^2}.$$

In analogy with the case of three-step walks, the derivative $W_4'(0;0)$ is particularly interesting because it is the Mahler measure of the multivariate polynomial $1 + x_1 + x_2 + x_3$. $\diamond$

**Example 3.24. (Second derivative of $W_4(\nu;\cdot)$ in even dimensions)** In [BSWZ12, (6.5), (6.10)] evaluations for the second derivatives $W_4''(0;0)$ and $W_4''(0;2)$ are given in terms of polylogarithmic constants. It follows from these evaluations and the functional equation (63) that the derivative values $W_4''(0;2k)$ all lie in the $\mathbb{Q}$-linear span of

$$1, \quad \pi^2, \quad \log^2 2, \quad \frac{1}{\pi^2}, \quad \frac{\log 2}{\pi^2}, \quad \frac{\zeta(3)}{\pi^2}, \quad \frac{\log^4 2}{\pi^2}, \quad \frac{\zeta(3)\log 2}{\pi^2}, \quad \frac{\mathrm{Li}_4(1/2)}{\pi^2},$$

where $\mathrm{Li}_n(z) := \sum_{k\geqslant 1} z^k/k^n$ is the polylogarithm of order $n$. Indeed, we realize from the dimensional recursion (65) that the same is true for the values $W_4''(\nu;2k)$ in all even dimensions. For example,

$$\begin{aligned} W_4''(1;0) &= \frac{253}{432} + \frac{5W_4'(0;0)}{162} - \frac{239W_4'(0;2)}{648} - \frac{26W_4''(0;0)}{27} + \frac{53W_4''(0;2)}{108} \\ &= -\frac{25}{48} - \frac{1}{5}\pi^2 - \log^2 2 + \frac{1193}{54\pi^2} - \frac{106}{3}\frac{\log 2}{\pi^2} + \frac{21}{4}\frac{\zeta(3)}{\pi^2} \\ &\quad + \frac{\log^4 2}{\pi^2} + 21\frac{\zeta(3)\log 2}{\pi^2} + 24\frac{\mathrm{Li}_4(1/2)}{\pi^2}. \end{aligned}$$

The number of (presumed) transcendental constants can be somewhat reduced when working in terms of Kummer-type polylogarithms, as illustrated in [BS12, Theorem 4.7]. ◇

**Example 3.25. (Ordinary generating function for even moments with four steps)** In (62) we noted that the ordinary generating function of the moments $W_4(0;2k)$ has a concise hypergeometric expression. It is natural to wonder if this result extends to higher dimensions.

Combining the recursive relations (63) and (65), we are able to derive that the ordinary generating function of the moments $W_4(\nu;2k)$, when complemented with an appropriate principal part (as in Theorem 3.6), can be obtained from the corresponding generating function of $W_4(\nu-1;2k)$ as well as its first two derivatives. Because the precise relationship is not too pleasant, we only record the simplified generating function,

$$-\frac{1}{2x^2} + \frac{1}{x} + \sum_{n=0}^{\infty} W_4(1;2k)x^k = (32x-7)F_0^2 - (4x-1)\left[(32x+3)F_0F_1 - \left(16x^2+10x+\frac{1}{4}\right)F_1^2\right], \tag{66}$$

that we obtain in dimension 4. Here,

$$F_\lambda := \frac{1}{2\cdot 3^\lambda x(16x-1)^{1-\lambda}}\frac{\mathrm{d}^\lambda}{\mathrm{d}x^\lambda}\,{}_2F_1\left(\begin{matrix}\frac{1}{6},\frac{1}{3}\\1\end{matrix}\middle|\frac{108x}{(16x-1)^3}\right),$$

which, by the differentiation formula [DLMF, (16.3.1)]

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,{}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\middle|x\right) = \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,{}_pF_q\left(\begin{matrix}a_1+1,\ldots,a_p+1\\b_1+1,\ldots,b_q+1\end{matrix}\middle|x\right), \tag{67}$$

can be expressed as generalized hypergeometric functions. It would be nice if it was possible to make the general case as explicit as we did in Theorem 3.6 for three-step walks, but we have not succeeded in doing so. ◇

## 3.4 Moments of 5-step walks

As in the planar case, as well as in many related problems, it is much harder to obtain explicit results in the case of five or more steps. This is reflected, for instance, in the fact that an application of Corollary 2.21 with $n_1 = 3$ and $n_2 = 2$, and appealing to (39), results in a double (and not single) sum of hypergeometric terms.

**Lemma 3.26.** *The nonnegative even moments for a* 5*-step walk in* $d$ *dimensions are*

$$\begin{aligned} W_5(\nu;2k) &= \sum_{j=0}^{k}\binom{k}{j}\frac{\binom{k+\nu}{j}}{\binom{j+\nu}{j}}\frac{\binom{2(k-j+\nu)}{k-j}}{\binom{k-j+\nu}{k-j}}W_3(\nu;2j) \\ &= \sum_{j=0}^{k}\binom{k}{j}\frac{\binom{k+\nu}{j}}{\binom{j+\nu}{j}}\frac{\binom{2(k-j+\nu)}{k-j}}{\binom{k-j+\nu}{k-j}}\sum_{i=0}^{j}\binom{j}{i}\frac{\binom{j+\nu}{i}\binom{2i+2\nu}{i}}{\binom{i+\nu}{i}^2}. \end{aligned} \tag{68}$$

**Example 3.27.** As in the case of three and four steps, we can apply creative telescoping to the binomial sum (68) to derive a recursion for the moments $W_5(\nu;2k)$. In contrast to the three-term recursions (48) and (63), we now obtain a four-term recursion, namely

$$\begin{aligned} &(k+2\nu+1)(k+3\nu+1)(k+4\nu+1)(k+5\nu+1)W_5(\nu;2k+2) \\ &= a\left(\nu;k+\tfrac{1}{2}\right)W_5(\nu;2k) - b(\nu;k)W_5(\nu;2k-2) + 225k(k-1)(k+\nu)(k-1+\nu)W_5(\nu;2k-4), \end{aligned} \tag{69}$$

where

$$\begin{aligned} a(\nu;m) &:= 35m^4 + 350\nu m^3 + \left(1183\nu^2 + \tfrac{21}{2}\right)m^2 \\ &\quad + \left(1540\nu^2 + \tfrac{105}{2}\right)\nu m + \left(600\nu^4 + \tfrac{237}{4}\nu^2 + \tfrac{3}{16}\right), \\ b(\nu;k) &:= k(k+\nu)(259k^2 + 1295k\nu + 1450\nu^2 + 26). \end{aligned}$$

As in Example 3.8, $W_5(\nu;s)$ is analytic and suitably bounded for Re $s \geqslant 0$, so that we may conclude from Carlson's Theorem 3.7 that the recursion (69) extends to complex $k$. $\diamond$

**Example 3.28. (Dimensional recursion for $W_5(\nu;s)$)** Creative telescoping, applied to the binomial sum (68) for $W_5(\nu;2k)$, allows us to derive the following more involved counterpart of the dimensional recursions (52), (65) for three and four steps.

$$\begin{aligned} 0 &= 3(s+2)(s+4)(s+2\nu+2)(s+8\nu)(s+10\nu-2)(s+10\nu)W_5(\nu;s) \\ &\quad -450\nu^4(s+4)(s+2\nu+2)W_5(\nu-1;s+2) \\ &\quad +4\nu^4 a(\nu;s)W_5(\nu-1;s+4) - 2\nu^4 b(\nu;s)W_5(\nu-1;s+6), \end{aligned} \tag{70}$$

where

$$\begin{aligned} a(\nu;s) &:= 107s^2 + 2(445\nu+152)s + 2(550\nu^2 + 1165\nu - 78), \\ b(\nu;s) &:= (s+4\nu+2)(13s+110\nu-16). \end{aligned}$$

This recursion is first obtained for nonnegative even integers $s$, and then extended to complex values using Carlson's Theorem 3.7. $\diamond$

# 4 Densities of short random walks

## 4.1 Densities of 2-step walks

We find an explicit formula for the probability density $p_2(\nu;x)$ of the distance to the origin after two random steps in $\mathbb{R}^d$ by computing the Bessel integral (6). An equivalent formula is given in [Wan13, Corollary 4.2], which exploited the fact that the probability density is essentially the inverse Mellin transform of the moments which are evaluated in Theorem 3.2.

**Lemma 4.1.** *The probability density function of the distance to the origin in $d \geqslant 2$ dimensions after 2 steps is, for $0 < x < 2$,*

$$p_2(\nu; x) = \frac{2}{\pi \binom{2\nu}{\nu}} x^{2\nu} (4 - x^2)^{\nu - 1/2}. \tag{71}$$

*Proof.* It follows from (17) that

$$p_2(\nu; x) = \nu! 2^\nu x^{\nu+1} \int_0^\infty t^{1-\nu} J_\nu(tx) J_\nu^2(t) \mathrm{d}t.$$

From [Wat41, Chapter 13.46], we have the integral evaluation

$$\int_0^\infty t^{1-\nu} J_\nu(at) J_\nu(bt) J_\nu(ct) \mathrm{d}t = \frac{2^{\nu-1} \Delta^{2\nu-1}}{(abc)^\nu \Gamma(\nu + 1/2) \Gamma(1/2)}, \tag{72}$$

assuming that $\mathrm{Re}\,(\nu) > -1/2$ and that $a, b, c$ are the sides of a triangle of area $\Delta$. In our case,

$$\Delta = \tfrac{x}{2} \sqrt{1 - (\tfrac{x}{2})^2},$$

and therefore

$$p_2(\nu; x) = \frac{\nu!}{\Gamma(\nu + 1/2) \Gamma(1/2)} x^{2\nu} \left(1 - (\tfrac{x}{2})^2\right)^{\nu - 1/2},$$

which is equivalent to (71). □

Note that (71) reflects the general fact, discussed in Section 2.2, that the densities $p_n(\nu; x)$ are piecewise polynomial in odd dimensions.

For comparison with the case of 3 steps, we record that the density $p_2(\nu; x)$ satisfies the following functional equation: if $F(x) := p_2(\nu; x)/x$, then

$$F(x) = F\left(\sqrt{4 - x^2}\right). \tag{73}$$

**Remark 4.2.** The probability density in two dimensions, that is

$$p_2(0; x) = \frac{2}{\pi \sqrt{4 - x^2}},$$

is readily identified as the distribution of $2|\cos\theta|$ with $\theta$ uniformly distributed on $[0, 2\pi]$. In other words, if $\boldsymbol{X} = (X_1, X_2)$ is uniformly distributed on the sphere of radius 2 in $\mathbb{R}^2$, then $p_2(0; x)$ describes the distribution of $|X_1|$. It is then natural to wonder whether this observation extends to higher dimensions.

Indeed, if $\boldsymbol{X} = (X_1, X_2)$ is uniformly distributed on the sphere of radius 2 in $\mathbb{R}^{2d-2}$, where $\boldsymbol{X}$ is partitioned into $X_1, X_2 \in \mathbb{R}^{d-1}$, then $p_2(\nu; x)$ describes the distribution of $|X_1|$. Details are left to the interested reader. In this stochastic interpretation, the invariance of (73) under $x \mapsto \sqrt{4 - x^2}$ is a reflection of the fact that $|X_1|$ and $|X_2| = \sqrt{4 - |X_1|^2}$ share the same distribution.

## 4.2 Densities of 3-step walks

It was shown in [BSWZ12] that the density $p_3(0;x)$ of the distance to the origin after three random steps in the plane has the closed form

$$p_3(0;x) = \frac{2\sqrt{3}}{\pi}\frac{x}{3+x^2}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\right|\frac{x^2(9-x^2)^2}{(3+x^2)^3}\right), \tag{74}$$

valid on the interval $(0,3)$. We next generalize this hypergeometric expression to arbitrary dimensions. In order to do so, we need to establish the behaviour of $p_3(\nu;x)$ at the end points.

To begin with, we use information on the pole structure of the moments $W_3(\nu;s)$ to deduce the asymptotic behaviour of $p_3(\nu;x)$ as $x \to 0^+$.

**Proposition 4.3.** *For positive half-integer $\nu$ and $x \to 0^+$,*

$$p_3(\nu;x) \sim \frac{2}{\sqrt{3}\pi}\frac{3^\nu}{\binom{2\nu}{\nu}}x^{2\nu+1}. \tag{75}$$

*Proof.* As shown in Corollary 2.15, the moments $W_3(\nu;s)$ are analytic for Re $s > -d$ and the first pole has residue

$$\operatorname{Res}_{s=-d} W_3(\nu;s) = \nu!2^\nu \int_0^\infty x^{1-\nu} J_\nu^3(x)\mathrm{d}x = \frac{2}{\sqrt{3}\pi}\frac{3^\nu}{\binom{2\nu}{\nu}},$$

where the last equality is another special case of (72). The asymptotic behaviour (75) then follows because $W_3(\nu;s-1)$ is the Mellin transform of $p_3(\nu;x)$. □

On the other hand, to obtain the asymptotic behaviour of $p_3(\nu;x)$ as $x \to 3^-$, we have to work a bit harder than in the case of the behaviour (75) as $x \to 0^+$.

**Proposition 4.4.** *For positive half-integer $\nu$ as $x \to 3^-$,*

$$p_3(\nu;x) \sim \frac{\sqrt{3}}{2\pi}\frac{2^{2\nu}3^\nu}{\binom{2\nu}{\nu}}(3-x)^{2\nu}. \tag{76}$$

*Proof.* Using Theorem 2.9 together with the fact that, for $x \in [0,2]$,

$$p_2(\nu;x) = \frac{2}{\pi\binom{2\nu}{\nu}}x^{2\nu}(4-x^2)^{\nu-1/2},$$

we find

$$p_3(\nu;x) = \frac{(2x)^{2\nu+1}}{\binom{2\nu}{\nu}^2\pi^2}\int_{-1}^{\min(1,\frac{3-x^2}{2x})}\frac{(4-(1+2\lambda x+x^2))^{\nu-1/2}}{(1+2\lambda x+x^2)^{1/2}}(1-\lambda^2)^{\nu-1/2}\mathrm{d}\lambda.$$

Observe that the upper bound of integration is 1 if $x \in [0,1]$, and $(3-x^2)/(2x)$ for $x \in [1,3]$. On $x \in [1,3]$, after substituting $t = \sqrt{1+2\lambda x+x^2}/x$, we therefore find

$$p_3(\nu;x) = \frac{(2x)^2}{\binom{2\nu}{\nu}^2\pi^2}\int_{1-\frac{1}{x}}^{\frac{2}{x}}\{(t^2x^2-4)(t^2x^2-(x+1)^2)(t^2x^2-(x-1)^2)\}^{\nu-1/2}\mathrm{d}t.$$

Note that the polynomial in the integrand factors into $(2-tx)(tx-(x-1))$ times a factor which approaches 192 as $x \to 3^-$ and $t \to 2/3$. Hence, as $x \to 3^-$,

$$p_3(\nu; x) \sim \frac{(2x)^2 \cdot 192^{\nu-1/2}}{\binom{2\nu}{\nu}^2 \pi^2} \int_{1-\frac{1}{x}}^{\frac{2}{x}} \{(2-tx)(tx-(x-1))\}^{\nu-1/2} \mathrm{d}t.$$

We now relate this integral to the incomplete beta function to find

$$\int_{1-\frac{1}{x}}^{\frac{2}{x}} \{(2-tx)(tx-(x-1))\}^{\nu-1/2} \mathrm{d}t = \binom{2\nu}{\nu} \frac{\pi(3-x)^{2\nu}}{2^{4\nu} x}.$$

Putting these together, we conclude that (76) holds. $\square$

We are now in a position to generalize (74) to higher dimensions.

**Theorem 4.5. (Hypergeometric form for $p_3$)** *For any half-integer $\nu \geqslant 0$ and $x \in [0,3]$, we have*

$$\frac{p_3(\nu; x)}{x} = \frac{2\sqrt{3}}{\pi} \frac{3^{-3\nu}}{\binom{2\nu}{\nu}} \frac{x^{2\nu}(9-x^2)^{2\nu}}{3+x^2} {}_2F_1\left( \begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1+\nu \end{matrix} \middle| \frac{x^2(9-x^2)^2}{(3+x^2)^3} \right). \tag{77}$$

*Proof.* We observe that both sides of (77) satisfy the differential equation $A_3 \cdot y(x) = 0$, where the differential operator $A_3$ is given by

$$x(x^2-9)(x^2-1)D_x^2 - (2\nu-1)(5x^4-30x^2+9)D_x + 4x(x^2-3)(3\nu-1)(2\nu-1),$$

where $D_x = \frac{\mathrm{d}}{\mathrm{d}x}$. Note that, in the case of $p_3(\nu; x)$, this is a consequence of the functional equation resulting from (48). The indices of this differential equation at $x=0$ are 0 and $2\nu$. It follows from Proposition 4.3 that $p_3(\nu; x)/x$ is the unique solution $y(x)$, on $(0,1)$, such that

$$y(x) \sim \frac{2}{\sqrt{3}\pi} \frac{3^\nu}{\binom{2\nu}{\nu}} x^{2\nu}$$

as $x \to 0^+$. Since this property is satisfied by the right-hand side of (77) as well, it follows that (77) holds for all $x \in (0,1)$.

Similarly, to show (77) for all $x \in (1,3)$, we use the fact that the differential equation has indices 0 and $2\nu$ at $x=3$ as well. In light of Proposition 4.4, $p_3(\nu; x)/x$ is the unique solution $y(x)$, on $(1,3)$, such that, as $x \to 3^-$,

$$y(x) \sim \frac{1}{2\sqrt{3}\pi} \frac{2^{2\nu} 3^\nu}{\binom{2\nu}{\nu}} (3-x)^{2\nu}.$$

Again, it is routine to verify that this property is also satisfied by the right-hand side of (77). Hence, it follows that (77) holds for all $x \in (1,3)$. $\square$

As a consequence, we have the following functional equation for the probability density function $p_3(\nu; x)$. The role of the involution $x \mapsto \sqrt{4-x^2}$ for 2 steps, see (73), is now played by the involution $x \mapsto \frac{3-x}{x+1}$.

**Corollary 4.6. (Functional equation for $p_3$)** *For any half-integer $\nu \geqslant 0$ and $x \in [0,3]$, the function $F(x) := p_3(\nu;x)/x$ satisfies the functional equation*

$$F(x) = \left(\frac{1+x}{2}\right)^{6\nu-2} F\left(\frac{3-x}{1+x}\right). \tag{78}$$

*Proof.* The hypergeometric right-hand side of (77) clearly satisfies the functional equation (78). ☐

It would be very interesting to have a probabilistic interpretation of this functional equation satisfied by the densities $p_3(\nu;x)$.

**Remark 4.7.** We note the following relation between the functional equations for the two- and three-step case in (73) and (78), respectively. Namely, if

$$X = \frac{x+y}{2} + 1, \quad Y = \frac{x-y}{2i},$$

then the relation $y = \frac{3-x}{1+x}$ translates into $Y^2 = 4 - X^2$. It is natural to wonder whether this observation might help explain the functional equation (78) for three-step densities.

**Example 4.8.** As a consequence of the contiguity relations satisfied by hypergeometric functions, we derive from (77) that the densities $p_3(\nu;x)$ satisfy the dimensional recursion

$$\begin{aligned} p_3(\nu+1;x) \quad = \quad & \tfrac{\nu(\nu+1)^2}{6(2\nu+1)(3\nu+1)(3\nu+2)}(x^2-3)(x^2-6x-3)(x^2+6x-3)p_3(\nu;x) \\ & + \tfrac{\nu^2(\nu+1)^2}{12(2\nu-1)(2\nu+1)(3\nu+1)(3\nu+2)}x^2(x^2-1)^2(x^2-9)^2 p_3(\nu-1;x), \end{aligned}$$

where $\nu \geqslant 1$. ◇

**Example 4.9.** It follows from the hypergeometric formula (77) that the densities $p_3(\nu;x)$, for $\nu > 0$, take the special values

$$p_3(\nu;1) = \frac{3}{4\pi^2}\frac{2^{6\nu}}{\nu}\frac{(\nu!)^5}{(2\nu)!(3\nu)!}.$$

In particular, $p_3(\nu;1) \in \mathbb{Q}$ in odd dimensions, and $p_3(\nu;1) \in \mathbb{Q} \cdot \frac{1}{\pi^2}$ in even dimensions. ◇

**Example 4.10.** From (19) and the functional equation of Corollary 4.6, for $\nu > 1$, we learn that

$$p_3'''(\nu;1) = \frac{9}{2}\nu p_3''(\nu;1) - \frac{3}{8}(3\nu-1)(6\nu^2+5\nu+2)p_3(\nu;1),$$

but learn nothing about $p_3''(\nu;1)$. ◇

### 4.3 Densities of 4-step walks

It is shown in [BSWZ12] that, in the planar case, the probability density of the distance to the origin after four steps admits the hypergeometric closed form

$$p_4(0;x) = \frac{2}{\pi^2}\frac{\sqrt{16-x^2}}{x} \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{matrix}\right| \frac{(16-x^2)^3}{108x^4}\right). \tag{79}$$

In this section we obtain a higher-dimensional analog of this evaluation by demonstrating that hypergeometric formulae can be given for the densities $p_4(\nu;x)$ in all even dimensions.

**Example 4.11.** Since the Mellin transform of the density $p_4(\nu;x)$ is given by the corresponding probability moments $W_4(\nu;s-1)$, the recursion (63) for the moments translates into a differential equation for the density. We refer to [BSWZ12] for details. We thus find that $p_4(\nu;x)/x$ is annihilated by the differential operator

$$\begin{aligned}&x^3(x^2-16)(x^2-4)D_x^3-3(2\nu-1)x^2(3x^4-40x^2+64)D_x^2\\&+(2\nu-1)x((52\nu-19)x^4-(416\nu-152)x^2+256\nu-64)D_x\\&-8(4\nu-1)(3\nu-1)(2\nu-1)x^2(x^2-4),\end{aligned}\tag{80}$$

where $D_x=\frac{\mathrm{d}}{\mathrm{d}x}$. We conclude that, in the planar case, $p_4(0;x)$ satisfies the differential equation

$$\begin{aligned}0 \quad = \quad & x^3(x^2-16)(x^2-4)p_4'''(0;x)+6x^2(x^4-10x^2)p_4''(0;x)\\&+x(7x^4-32x^2+64)p_4'(0;x)+(x^4-64)p_4(0;x),\end{aligned}\tag{81}$$

which agrees (up to a typo there) with [BSWZ12, (2.7)].

The differential equation (81) is the symmetric square of a second order differential equation and, moreover, admits modular parametrization [BSWZ12, Remark 4.11]. These ingredients ultimately lead to the hypergeometric closed form (79). The differential equation associated with (80), on the other hand, is a symmetric square of a second order differential equation only in the cases $\nu=0$ and $\nu=1/2$. ◇

**Example 4.12.** In three dimensions, that is when $\nu=1/2$, the density is

$$p_4(1/2;x)=\frac{x}{16}(8-x^2+2(x-2)|x-2|),\tag{82}$$

for $x\in[0,4]$. This is equivalent to (24) and may be derived directly from (6) or from Theorem 2.11. ◇

**Example 4.13.** Basic Mellin calculus connects the asymptotic behaviour of $p_n(\nu;x)$ as $x\to 0^+$ with the nature of the poles of $W_n(\nu;s)$ in the left half-plane. For instance, from the explicit information in Example 2.17 on the poles of $W_4(1;s)$ at $s=-4$ and $s=-6$, we conclude that

$$p_4(1;x)=\frac{4}{\pi^2}x^3+\left(\frac{1}{16\pi^2}-\frac{9\log(2)}{4\pi^2}\right)x^5+\frac{3}{4\pi^2}\log(x)x^5+O(x^7)$$

as $x\to 0^+$. ◇

The following result connects the 4-step density in $d$ dimensions with the corresponding density in $d-2$ dimensions. In particular, using (79) and (82) as base cases, this provides a way to obtain explicit formulas for the densities $p_4(\nu;x)$ in all dimensions.

**Theorem 4.14. (Dimensional recursion for $p_4$)** *For $0\leqslant x\leqslant 4$ and any half-integer $\nu\geqslant 0$,*

$$\begin{aligned}&\frac{3(2\nu+1)(3\nu+1)(3\nu+2)(4\nu+1)(4\nu+3)}{64(\nu+1)^3}p_4(\nu+1;x)\\=\quad&-a(\nu;x^2/8)p_4(\nu;x)+xb(\nu;x^2/8)p_4'(\nu;x)-c(\nu;x^2/8)p_4''(\nu;x),\end{aligned}\tag{83}$$

*where*

$$\begin{aligned} a(\nu;x) &:= (4\nu-1)(6\nu-1)x^4 + 2(100\nu^2-23\nu-1)x^3 + 2(2\nu+3)(12\nu+1)x^2 \\ &\quad -2(60\nu^2+13\nu+1)x + (2\nu+1)(4\nu+1), \\ b(\nu;x) &:= (10\nu-3)x^4 + 5(12\nu-1)x^3 - \tfrac{21}{2}(8\nu-1)x^2 - 20\nu x + 6\nu + 1, \\ c(\nu;x) &:= 4x(x-2)(2x-1)(x^2+5x+1). \end{aligned}$$

*Proof.* We have already observed in (63) and (65) that the moments $W_4(\nu;s)$ satisfy a functional equation connecting $W_4(\nu;s)$, $W_4(\nu;s+2)$, $W_4(\nu;s+4)$, as well as a functional equation relating $W_4(\nu+1;s)$, $W_4(\nu;s+2)$, $W_4(\nu;s+4)$. The usual Mellin calculus translates the first of these two into the third-order differential equation recorded in (80), while the second is translated into a more complicated equation involving seven terms ranging from $p_4(\nu;x)$ to $p_4^{(4)}(\nu+1;x)$. With assistance of the *Mathematica* package `HolonomicFunctions`, which accompanies Koutschan's thesis [Kou09], we compute a Gröbner basis for the ideal that these two relations generate and use it to find the claimed relation involving $p_4(\nu+1;x)$ as well as $p_4(\nu;x)$ and its first two derivatives. ☐

**Example 4.15. (Hypergeometric form for $p_4(1;x)$)** By combining (79) with Theorem 4.14 and (67), we conclude, for instance, that the 4-step density in four dimensions, for $x \in (2,4)$, can be hypergeometrically represented as

$$p_4(1;x) = \frac{(16-x^2)^{5/2}}{(24\pi)^2 x}\left[-(x^2+8)^2 G_0 + \frac{r(x^2)}{7!!\,x^4}G_1 + \frac{(16-x^2)^3 s(x^2)}{13!!\,(2/3)^4 x^8}G_2\right], \tag{84}$$

with $(2n+1)!! = (2n+1)(2n-1)\cdots 3\cdot 1$ and

$$G_\lambda := {}_3F_2\left(\begin{matrix}\frac12+\lambda, \frac12+\lambda, \frac12+\lambda \\ \frac56+\lambda, \frac76+\lambda\end{matrix}\middle|\frac{(16-x^2)^3}{108x^4}\right)$$

as well as

$$\begin{aligned} r(x) &:= x^5 + 55x^4 + 1456x^3 + 25664x^2 - 90112x - 262144, \\ s(x) &:= (x-4)(x+32)^2(x^2+40x+64). \end{aligned}$$

As in (79), taking the real part of the hypergeometric functions provides a formula for $p_4(1;x)$ which is valid for $x \in (0,2)$ as well. The above hypergeometric formula also provides a modular parametrization (in a suitably generalized sense) for $p_4(1;x)$. This is a consequence of

$$p_4(0;8ix(\tau)) = \frac{6(2\tau+1)}{\pi}f(\tau),$$

which is proved in [BSWZ12, (4.16)] and which involves the modular function

$$x(\tau) = \frac{\eta(2\tau)^3\eta(6\tau)^3}{\eta(\tau)^3\eta(3\tau)^3}$$

and the weight 2 modular form

$$f(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau).$$

Differentiating this modular parametrization of $p_4(0;x)$, we find that $p_4(1;8ix(\tau))$ can be expressed in terms of modular quantities such as $f'(\tau)/x'(\tau)$. Then inductively we obtain the like result for higher even dimensions. ◇

**Example 4.16. ($p_4(\nu;2)$ in even dimensions)** Motivated by [BSWZ12, Corollary 4.8], which proves that

$$p_4(0;2) = \frac{2^{7/3}\pi}{3\sqrt{3}}\Gamma\left(\frac{2}{3}\right)^{-6} = \frac{\sqrt{3}}{\pi}W_3(0;-1),$$

or $\pi/\sqrt{3}p_4(0;2) = W_3(0;-1)$, we discover that, for integers $\nu \geqslant 0$, $\pi/\sqrt{3}p_4(\nu;2)$ is a rational combination of the moments $W_3(0;-1)$ and $W_3(0;1)$. For instance,

$$\frac{\pi}{\sqrt{3}}p_4(1;2) = -\frac{14}{3}W_3(0;-1) + \frac{10}{3}W_3(0;1)$$

and

$$\frac{\pi}{\sqrt{3}}p_4(2;2) = \frac{6656}{315}W_3(0;-1) - \frac{704}{63}W_3(0;1).$$

To deduce the first of these from equation (83) takes a little care as $p_4(1;x)$ is not differentiable at 2 and one must take the limit from the left in (84); likewise for higher dimensions (note that $c(\nu;2) = 0$). Theorem 4.14 specializes to

$$\begin{aligned} & (2\nu+1)(3\nu+1)(3\nu+2)(4\nu+1)(4\nu+3)p_4(\nu+1;2) \\ = \;& 4(\nu+1)^3[(6\nu+1)(12\nu-7)p_4(\nu;2) - 30(6\nu-1)p_4'(\nu;2)]. \end{aligned}$$

On the other hand, specializing a corresponding relation among $p_4(\nu;x)$, $p_4(\nu+1;x)$, $p_4(\nu+2;x)$ and $p_4'(\nu;x)$, which one obtains as in the proof of Theorem 4.14, we find

$$\begin{aligned} & (2\nu-1)(2\nu+1)(3\nu-2)(3\nu-1)(3\nu+1)(3\nu+2)(4\nu+1)(4\nu+3)p_4(\nu+1;2) \\ = \;& -16(\nu+1)^3(2\nu-1)(3\nu-2)(3\nu-1)(54\nu^2+1)p_4(\nu;2) \\ & +576\nu^3(\nu+1)^3(6\nu-5)(6\nu-1)p_4(\nu-1;2). \end{aligned}$$

◇

## 4.4 Densities of 5-step walks

The 5-step densities for dimensions up to 9 are plotted in Figure 4.

A peculiar feature of the planar density is its striking (approximate) linearity on the initial interval $[0,1]$. This phenomenon was already observed by Pearson [Pea06], who commented on $p_5(0;x)/x$, between $x=0$ and $x=1$, as follows:

> "the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a *straight* line... Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of $J$ products [that is, (6)] to give extremely close approximations to such simple forms as horizontal lines."

Pearson's comment was revisited by Fettis [Fet63], who rigorously established the nonlinearity. In [BSWZ12, Theorem 5.2], it is shown that the density satisfies a certain fourth-order differential equation, and that, for small $x > 0$,

$$p_5(0;x) = \sum_{k=0}^{\infty} r_{5,k}\, x^{2k+1}, \tag{85}$$

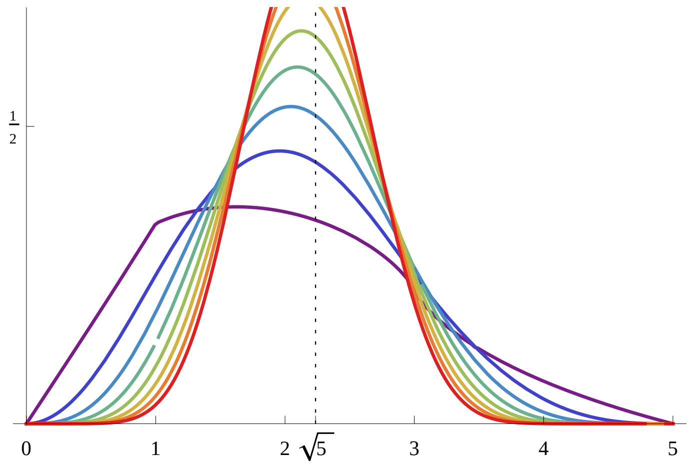


Figure 4: $p_5(\nu; x)$ for $\nu = 0, \frac{1}{2}, 1, \ldots, \frac{7}{2}$

where

$$\begin{aligned} & (15(2k+2)(2k+4))^2 r_{5,k+2} \\ = \; & (259(2k+2)^4 + 104(2k+2)^2) r_{5,k+1} \\ & -(35(2k+1)^4 + 42(2k+1)^2 + 3) r_{5,k} + (2k)^4 r_{5,k-1}, \end{aligned} \tag{86}$$

with initial conditions $r_{5,-1} = 0$ and

$$r_{5,k} = \operatorname{Res}_{s=-2k-2} W_5(0; s).$$

Numerically, we thus find that

$$p_5(0; x) = 0.329934\, x + 0.00661673\, x^3 + 0.000262333\, x^5 + O(x^7),$$

which reflects the approximate linearity of $p_5(0; x)$ for small $x$.

By (18), the residue $r_{5,0} = p_5'(0; 0)$ can be expressed as $p_4(0; 1)$. The modularity of $p_4$ in the planar case, combined with the *Chowla–Selberg formula* [SC67], then permits us to obtain the explicit formula [BSWZ12, Theorem 5.1]

$$r_{5,0} = \frac{\sqrt{5}}{40} \frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\pi^4}. \tag{87}$$

Moreover, high-precision numerical calculations lead to the conjectural evaluation [BSWZ12, (5.3)]

$$r_{5,1} = \frac{13}{225} r_{5,0} - \frac{2}{5\pi^4} \frac{1}{r_{5,0}}, \tag{88}$$

and the recursion (86) then implies that all the coefficients $r_{5,k}$ in (85) can be expressed in terms of $r_{5,0}$.

**Theorem 4.17. (Residues of $W_5(0; s)$)** *The conjectural relation* (88) *is true.*

*Proof.* As noted after Corollary 2.15, we have

$$\operatorname{Res}_{s=-4} W_5(1; s) = p_4(1; 1).$$

On the other hand, applying the dimensional recursion (70) for $W_5(\nu; s)$ with $\nu = 1$ and using the values $W_5(0; 0) = 1$ and $W_5(0; 2) = 5$, we obtain

$$\operatorname{Res}_{s=-4} W_5(1; s) = \frac{7}{6} - \frac{25}{32} r_{5,0} + \frac{35}{24} W_5'(0; 0) - \frac{7}{24} W_5'(0; 2),$$

since $W_5(0; s)$ has simple poles only [BSW13, Example 2.5]. As noted in [BSWZ12, (6.2)], it is a consequence of the functional equation (69) that

$$225 r_{5,1} = 26 r_{5,0} - 16 - 20 W_5'(0; 0) + 4 W_5'(0; 2).$$

Combining the last three equations, we arrive at

$$p_4(1; 1) = \frac{107}{96} r_{5,0} - \frac{525}{32} r_{5,1}.$$

Equation (88) is therefore equivalent to

$$p_4(1; 1) = \frac{1}{6} r_{5,0} + \frac{105}{16\pi^4} \frac{1}{r_{5,0}}. \tag{89}$$

The equality (89) can now be deduced (at least in principle) from the hypergeometric formula for $p_4(1; x)$, made explicit in Example 4.15, the modular parametrization of $p_4(0; x)$ as well as the Chowla–Selberg formula [SC67]. □

In conclusion, we know that $p_5(0; x)$ has a Taylor expansion (85) at $x = 0$, which converges and gives its values in the interval $[0, 1]$. Moreover, we have a recursive description of the Taylor coefficients and know that they are all $\mathbb{Q}$-linear combinations of $r_{5,0}$ in (87) and $1/(\pi^4 r_{5,0})$. All of these statements carry over to the 5-step densities $p_5(\nu; x)$ in any even dimension. Since the details are unwieldy, we only sketch why this is so.

Recall that the moments $W_5(\nu; s)$ satisfy the recursive relations (69) and (70). Indeed, there is a third relation which connects $W_5(\nu; s)$, $W_5(\nu; s+2)$, $W_5(\nu+1; s)$, $W_5(\nu+1; s+2)$. As in the proof of Theorem 4.14, the Mellin transform translates these three recursive relations into (complicated) differential relations for the densities $p_5(\nu; x)$. Assisted, once more, by Koutschan's package `HolonomicFunctions` [Kou09], we compute a Gröbner basis for the ideal that these three differential relations generate. From there, we find that there exists, in analogy with Theorem 4.14, a relation

$$x^2 p_5(\nu + 1; x) = A p_5(\nu; x) + B p_5'(\nu; x) + C p_5''(\nu; x) + D p_5'''(\nu; x),$$

with $A, B, C, D$ polynomials of degrees $12, 13, 14, 15$ in $x$ (with coefficients that are rational functions in $\nu$).

We therefore conclude inductively that, for integers $\nu$, the density $p_5(\nu; x)$ has a Taylor expansion (85) at $x = 0$ whose Taylor coefficients lie in the $\mathbb{Q}$-span of $r_{5,0}$ in (87) and $1/(\pi^4 r_{5,0})$. It remains an open challenge, including in the planar case, to obtain a more explicit description of $p_5(\nu; x)$.

# 5 Conclusion

We have shown that quite delicate results are possible for densities and moments of walks in arbitrary dimensions, especially for two, three and four steps. We find it interesting that induction between dimensions provided methods to show Theorem 4.17, a result in the plane that we previously could not establish [BSWZ12]. We also should emphasize the crucial role played by computer experimentation and by computer algebra. One stumbling block is that currently *Mathematica*, and to a lesser degree *Maple*, struggle with computing various of the Bessel integrals to more than a few digits — thus requiring considerable extra computational effort or ingenuity.

We leave some open questions:

- The even moments $W_n(0; 2k)$ associated to a random walk in two dimensions have combinatorial significance. They count *abelian squares* [RS09] of length $2k$ over an alphabet with $n$ letters (i.e., strings $xx'$ of length $2k$ from an alphabet with $n$ letters such that $x'$ is a permutation of $x$). As observed in Example 2.22, the even moments $W_n(1; 2k)$ are positive integers as well and we have expressed them in terms of powers of the Narayana triangular matrix, whose entries count certain lattice paths. Does that give rise to an interpretation of the even four-dimensional moments themselves counting similar combinatorially interesting objects?

- As discussed in Example 3.17, in the case $\nu = 0$, the moments $W_4(\nu; 2k)$ are the Domb numbers, for which a clean hypergeometric generating function is known. Referring to Example 3.25, we wonder if it is possible to give a compact explicit hypergeometric expression for the generating function of the even moments $W_4(\nu; 2k)$, valid for all even dimensions, as we did in Theorem 3.6 for three-step walks.

- Verrill has exhibited [Ver04] an explicit recursion in $k$ of the even moments $W_n(0; 2k)$ in the plane. Combined with a result of Djakov and Mityagin [DM04], proved more directly and combinatorially by Zagier [BSWZ12, Appendix A], these recursions yielded insight into the general structure of the densities $p_n(0; x)$. For instance, as shown in [BSWZ12, Theorem 2.4], it follows that these densities are real analytic except at 0 and the positive integers $n, n-2, n-4, \ldots$ It would be interesting to obtain similar results for any dimension.

- By exhibiting recursions relating different dimensions, we have shown that the odd moments of the distances after three and four random steps in any dimension can all be expressed in terms of the constants arising in the planar case. Is it possible to evaluate these odd moments in a closed (hypergeometric) form which reflects this observation?

- In the plane, various other fragmentary modular results are (conjecturally) known for five and six step walks, see [BSWZ12, (6.11), (6.12)] for representations of $W_5'(0; 0)$ and $W_6'(0; 0)$, conjectured by Rodriguez-Villegas [BLRVD03], as well as a discussion of their relation to *Mahler measures*. Are more comprehensive results possible?

**Acknowledgements.** The third author would like to thank J. M. Borwein for his invitation to visit the CARMA center in September 2014; this research was initiated at this occasion. Thanks are due to Ghislain McKay and Corey Sinnamon who explored the even moments during visiting student fellowships at CARMA in early 2015.

## References


[AMV13] T. Amdeberhan, V. H. Moll, and C. Vignat. A probabilistic interpretation of a sequence related to Narayana polynomials. *Online Journal of Analytic Combinatorics*, 8, June 2013.

[BBBG08] D. H. Bailey, J. M. Borwein, D. M. Broadhurst, and L. Glasser. Elliptic integral representation of Bessel moments. *J. Phys. A: Math. Theory*, 41:5203–5231, 2008.

[Ber13] O. Bernardi. A short proof of Rayleigh's Theorem with extensions. *The American Mathematical Monthly*, 120(4):362–364, 2013.

[BB01] D. Borwein and J. M. Borwein. Some remarkable properties of sinc and related integrals. *Ramanujan J.*, 5:73–90, 2001.

[BNSW11] J. M. Borwein, D. Nuyens, A. Straub, and J. Wan. Some arithmetic properties of short random walk integrals. *The Ramanujan Journal*, 26(1):109–132, 2011.

[BS12] J. M. Borwein and A. Straub. Log-sine evaluations of Mahler measures. *J. Aust Math. Soc.*, 92(1):15–36, 2012.

[BSW13] J. M. Borwein, A. Straub, and J. Wan. Three-step and four-step random walk integrals. *Experimental Mathematics*, 22(1):1–14, 2013.

[BSWZ12] J. M. Borwein, A. Straub, J. Wan, and W. Zudilin. Densities of short uniform random walks (with an appendix by Don Zagier). *Canadian Journal of Mathematics*, 64(5):961–990, 2012.

[BLRVD03] D. Boyd, D. Lind, F. Rodriguez Villegas, and C. Deninger. The many aspects of Mahler's measure. Final report of 2003 Banff workshop. Available at http://www.birs.ca/workshops/2003/03w5035/report03w5035.pdf, 2003.

[Bro09] D. J. Broadhurst. Bessel moments, random walks and Calabi-Yau equations. Preprint, 2009.

[DM04] P. Djakov and B. Mityagin. Asymptotics of instability zones of Hill operators with a two term potential. *C. R. Math. Acad. Sci. Paris*, 339(5):351–354, 2004.

[DLMF] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.9 of 2014-08-29. Online companion to [OLBC10].

[Fet63] H. E. Fettis. On a conjecture of Karl Pearson. *Rider Anniversary Volume*, pages 39–54, 1963.

[GP12] R. García-Pelayo. Exact solutions for isotropic random flights in odd dimensions. *Journal of Mathematical Physics*, 53(10):103504, 2012.

[Hug95] B. D. Hughes. *Random Walks and Random Environments*, volume 1. Oxford University Press, 1995.

[Kin63] J. F. C. Kingman. Random walks with spherical symmetry. *Acta Mathematica*, 109(1):11–53, July 1963.

[Klu06] J. C. Kluyver. A local probability problem. *Nederl. Acad. Wetensch. Proc.*, 8:341–350, 1906.

[Kou09] C. Koutschan. *Advanced Applications of the Holonomic Systems Approach*. PhD thesis, RISC, Johannes Kepler University, Linz, Austria, September 2009.

[Mar83] O. I. Marichev. *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables*. Ellis Horwood Limited, Chichester, England, 1983.

[OLBC10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, New York, NY, 2010. Print companion to [DLMF].

[Pea05] K. Pearson. The problem of the random walk. *Nature*, 72(1866):294, 1905.

[Pea06] K. Pearson. A mathematical theory of random migration. In *Drapers Company Research Memoirs*, number 3 in Biometric Series. Cambridge University Press, 1906.

[Ray19] L. Rayleigh. On the problem of random vibrations, and of random flights in one, two, or three dimensions. *Philosophical Magazine Series 6*, 37(220):321–347, April 1919.

[RS09] L. B. Richmond and J. Shallit. Counting abelian squares. *The Electronic Journal of Combinatorics*, 16, 2009.

[Rog09] M. D. Rogers. New ${}_5F_4$ hypergeometric transformations, three-variable Mahler measures, and formulas for $1/\pi$. *Ramanujan Journal*, 18(3):327–340, 2009.

[SC67] A. Selberg and S. Chowla. On Epstein's zeta-function. *J. Reine Angew. Math.*, 227:86–110, 1967.

[Slo15] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences, 2015. Published electronically at `http://oeis.org`.

[Sta99] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.

[Tit39] E. Titchmarsh. *The Theory of Functions*. Oxford University Press, 2nd edition, 1939.

[Ver04] H. A. Verrill. Sums of squares of binomial coefficients, with applications to Picard-Fuchs equations. Preprint, 2004.

[Wan13] J. G. Wan. *Random walks, elliptic integrals and related constants*. PhD thesis, University of Newcastle, 2013.

[Wat41] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 2nd edition, 1941.

[Zei90] D. Zeilberger. A holonomic systems approach to special function identities. *Journal of Computational and Applied Mathematics*, 32(3):321–368, 1990.

Jonathan M. Borwein
CARMA, University of Newcastle, Australia
jonathan.borwein@newcastle.edu.au

Armin Straub
Department of Mathematics, University of Illinois at Urbana-Champaign, USA
astraub@illinois.edu

Christophe Vignat
Department of Mathematics, Tulane University, USA
LSS, Supelec, Université Paris Sud, France
cvignat@tulane.edu