Eulerian Numbers

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Outline

Abstract Definition

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$$\Delta \, sep \triangleq \; sepq\theta(q)^{24} \, sep \triangleq \; sepq \prod_{n \geqslant 1} (1 - q^n)^{24} = \sum_{n \geqslant 1} \tau(n) q^n.$$

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Example

Denote $\sigma \in S_n$ as $[\sigma(1), \ldots, \sigma(n)]$.

$[5, 1, 3, 4, 2]$ has 2 ascents

$[2, 3, 4, 1, 5]$ has 3 ascents

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The Eulerian number $\langle {n} \rangle$ is the number of permutations in S

Simple Properties

- Row Sums

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- Recurrence

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle$$

Eulerian Triangle

					1					
				1		1				
			1		4		1			
		1		11		11		1		
	1		26		66		26		1	
1		57		302		302		57		1

Note

The triangle starts

$$\left\langle {1 \atop 0} \right\rangle$$

$$\left\langle {2 \atop 0} \right\rangle \qquad \left\langle {2 \atop 1} \right\rangle$$

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&\vdots \\
(xD)^n\frac{1}{1-x} &= \frac{x}{(1-x)^{n+1}}\sum_{k=0}^{n-1}\left\langle {n \atop k} \right\rangle x^k
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Counting Points in Hypercubes

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$$\begin{array}{ccccc} i & \leq & j & \leq & k \\ i & \leq & k & < & j \\ j & < & i & \leq & k \\ j & \leq & k & < & i \\ k & < & i & \leq & j \\ k & < & j & < & i \end{array} \quad \rightsquigarrow x^3 = \binom{x+2}{3} + 4\binom{x+1}{3} + \binom{x}{3}$$

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$$x^3 = \binom{x+2}{3} + 4\binom{x+1}{3} + \binom{x}{3}$$

- Generally,

$$x^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}$$

Counting Points in Hypercubes (Sums of Powers)

Using

$$x^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}$$

and

$$\Delta_x \binom{x+k}{n} = \binom{x+k}{n-1}$$

we get

$$\sum_{x=0}^{N} x^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \sum_{x=0}^{N} \binom{x+k}{n} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{N+k+1}{n+1}$$

Occurrence in Probability Theory

X_j iid, uniformly distributed on $[0, 1]$.

$$\frac{1}{n!}\left\langle {n \atop k} \right\rangle = P\left(\sum_{j=1}^{n} X_j \in [k, k+1]\right)$$

Asymptotics

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$$

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$$\left\langle {n \atop 2} \right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$$

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$$\vdots$$

$$\left\langle {n \atop k} \right\rangle \sim (k+1)^n \quad \text{as } n \to \infty$$

Generating Functions

- Let $A_{n,k} = \left\langle {n \atop k+1} \right\rangle$.

$$1 + \sum_{k,n \geq 1} A_{n,k} \frac{x^n y^k}{n!} = \frac{1-y}{1-ye^{(1-y)x}}$$

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- Let $A_{[r,s]} = \left\langle {r+s+1 \atop r} \right\rangle$.

$$\sum_{r,s \geq 0} A_{[r,s]} \frac{x^r y^s}{(r+s+1)!} = \frac{e^x - e^y}{xe^y - ye^x}$$