Getting started

Either open the file logsine.m and click "Run Package" or use something like this:

```
In[1]:= << "~/docs/math/mathematica/logsine.m"
```

```
LsToLi: evaluating log-sine integrals in polylogarithmic terms
     accompanying the paper "Special values of generalized log-sine integrals"
     -- Jonathan M. Borwein, University of Newcastle
     -- Armin Straub, Tulane University
     -- Version 2.0 (2013/04/03)
```

General usage

Log-sine integrals can be expressed in terms of polylogarithms:

```
In[2]:= LsToLi[Ls[5, 1, Pi / 3]]
```

Out[2]= $\frac{3}{2}\pi\,\mathrm{Cl}\left[\{4\},\ \frac{\pi}{3}\right]+\frac{1}{12}\pi^2\,\mathrm{Zeta}[3]-\frac{19\,\mathrm{Zeta}[5]}{4}$

We can check this numerically:

```
In[3]:= Ls[5, 1, Pi / 3] // N
        LsToLi[Ls[5, 1, Pi / 3]] // N
        N[LsToLi[Ls[5, 1, Pi / 3]], 100]
```

```
Out[3]= 0.379072

Out[4]= 0.379072

Out[5]= 0.37907210601263943123183786143132411189150176092619714873699853465680010503213193012037333
         998861251025
```

Combinations of Nielsen-type polylogarithms may be simplified:

```
In[6]:= LiReduce[Gl[{3, 1}, Pi / 3]]
```

Out[6]= $-\frac{23\,\pi^4}{19\,440}$

```
In[7]:= LiReduce[Cl[{5, 1}, t]]
```

Out[7]= $\frac{1}{2}\pi\,\mathrm{Cl}[\{5\},\ t]-\frac{1}{2}t\,\mathrm{Cl}[\{5\},\ t]+2\,\mathrm{Cl}[\{6\},\ t]-\frac{1}{6}\pi^2\,t\,\mathrm{Zeta}[3]+\frac{1}{4}\pi\,t^2\,\mathrm{Zeta}[3]-\frac{1}{12}t^3\,\mathrm{Zeta}[3]-\frac{1}{2}\pi\,\mathrm{Zeta}[5]+\frac{1}{2}t\,\mathrm{Zeta}[5]$

Advanced options

In[8]:=
```
LiReduce[Cl[{4, 1, 1}, Pi / 3]]
LiReduce[Cl[{4, 1, 1}, Pi / 3], UseReductionTable → False]
```

Out[8]= $-\frac{1}{18}\pi^2\,\mathrm{Cl}\left[\{4\}, \frac{\pi}{3}\right] + 3\,\mathrm{Cl}\left[\{6\}, \frac{\pi}{3}\right] - \frac{11}{324}\pi^3\,\mathrm{Zeta}[3] - \frac{29}{108}\pi\,\mathrm{Zeta}[5]$

Out[9]= $\mathrm{Cl}\left[\{4, 1, 1\}, \frac{\pi}{3}\right]$

Comments on the reduction of Nielsen-type polylogarithms

In[10]:=
```
NielsenArgsOfWeight[w_, t_] := Table[{Join[{w - n + 1}, Table[1, {n - 1}]], t}, {n, 1, w - 1}];
NielsenArgsUpToWeight[w_, t_] := Flatten[Table[NielsenArgsOfWeight[n, t], {n, 2, w}], 1];
```

The irreducible Clausen values of argument pi/3 and weight at most 10:

In[12]:=
```
Select[Table[Cl @@ a, {a, NielsenArgsUpToWeight[10, Pi / 3]}], # === LiReduce[#] &]
```

Out[12]= $\left\{\mathrm{Cl}\left[\{2\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{4\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{6\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{8\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{6, 1, 1\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{10\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{8, 1, 1\}, \frac{\pi}{3}\right]\right\}$

The irreducible Glaisher values of argument pi/3 and weight at most 10:

In[13]:=
```
Select[Table[Gl @@ a, {a, NielsenArgsUpToWeight[10, Pi / 3]}], # === LiReduce[#] &]
```

Out[13]= $\left\{\mathrm{Gl}\left[\{4, 1\}, \frac{\pi}{3}\right], \mathrm{Gl}\left[\{6, 1\}, \frac{\pi}{3}\right], \mathrm{Gl}\left[\{7, 1\}, \frac{\pi}{3}\right], \mathrm{Gl}\left[\{8, 1\}, \frac{\pi}{3}\right], \mathrm{Gl}\left[\{9, 1\}, \frac{\pi}{3}\right]\right\}$

The Clausen values of argument pi/3 and weight at most 10 for which extra relations exist:

In[14]:=
```
Select[Table[Cl @@ a, {a, NielsenArgsUpToWeight[10, Pi / 3]}],
 LiReduce[#, UseReductionTable → False] === # && LiReduce[#] =!= # &]
```

Out[14]= $\left\{\mathrm{Cl}\left[\{4, 1, 1\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{5, 1, 1\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{7, 1, 1\}, \frac{\pi}{3}\right], \mathrm{Cl}\left[\{6, 1, 1, 1, 1\}, \frac{\pi}{3}\right]\right\}$

The Glaisher values of argument pi/3 and weight at most 10 for which extra relations exist:

In[15]:=
```
Select[Table[Gl @@ a, {a, NielsenArgsUpToWeight[10, Pi / 3]}],
 LiReduce[#, UseReductionTable → False] === # && LiReduce[#] =!= # &]
```

Out[15]= $\left\{\mathrm{Gl}\left[\{5, 1\}, \frac{\pi}{3}\right], \mathrm{Gl}\left[\{6, 1, 1, 1\}, \frac{\pi}{3}\right], \mathrm{Gl}\left[\{7, 1, 1, 1\}, \frac{\pi}{3}\right]\right\}$

Examples from "Special values of generalized log-sine integrals"

In[16]:=
```
Table[LsToLi[Ls[n, 0, Pi]], {n, 2, 8}]
```

Out[16]= $\left\{0, -\frac{\pi^3}{12}, \frac{3}{2}\pi\,\mathrm{Zeta}[3], -\frac{19\pi^5}{240}, \frac{5}{4}\pi^3\,\mathrm{Zeta}[3] + \frac{45}{2}\pi\,\mathrm{Zeta}[5], -\frac{275\pi^7}{1344} - \frac{45}{2}\pi\,\mathrm{Zeta}[3]^2, \frac{133}{32}\pi^5\,\mathrm{Zeta}[3] + \frac{315}{8}\pi^3\,\mathrm{Zeta}[5] + \frac{2835}{4}\pi\,\mathrm{Zeta}[7]\right\}$

In[17]:=
```
-Ls[4, 2, Pi] // LsToLi
```

Out[17]= $\frac{3}{2}\pi\,\text{Zeta}[3]$

In[18]:=
```
-Ls[5, 1, Pi] // N
-Ls[5, 1, Pi] // LsToLi
% // N
```

Out[18]= 0.494072

Out[19]= $-6\,\text{Li}[\{3, 1, 1\}, -1] - \frac{1}{4}\pi^2\,\text{Zeta}[3] + \frac{105\,\text{Zeta}[5]}{32}$

Out[20]= 0.494072

In[21]:=
```
-Ls[6, 1, Pi] // LsToLi
```

Out[21]= $-\frac{3\pi^6}{1120} - 18\,\text{Li}[\{5, 1\}, -1] + 24\,\text{Li}[\{3, 1, 1, 1\}, -1] + 3\,\text{Zeta}[3]^2$

In[22]:=
```
Ls[5, 2, 2 Pi] // LsToLi
```

Out[22]= $-\frac{13\pi^5}{45}$

In[23]:=
```
$Assumptions = 0 < τ < Pi;
Ls[4, 1, τ] // LsToLi
```

Out[24]= $\frac{\pi^4}{180} - \frac{\pi^2\tau^2}{8} + \frac{\pi\tau^3}{6} - \frac{\tau^4}{16} - 2\tau\,\text{Gl}[\{2, 1\}, \tau] - 2\,\text{Gl}[\{3, 1\}, \tau]$

In[25]:=
```
-Ls[4, 1, Pi / 3] // LsToLi
```

Out[25]= $\frac{17\pi^4}{6480}$

In[26]:=
```
Table[LsToLi[Ls[n, 0, Pi / 3]], {n, 2, 7}]
```

Out[26]= $\left\{\text{Cl}\left[\{2\}, \frac{\pi}{3}\right], -\frac{7\pi^3}{108}, \frac{9}{2}\text{Cl}\left[\{4\}, \frac{\pi}{3}\right] + \frac{1}{2}\pi\,\text{Zeta}[3], -\frac{1543\pi^5}{19440} + 6\,\text{Gl}\left[\{4, 1\}, \frac{\pi}{3}\right],\right.$
$\left.\frac{135}{2}\text{Cl}\left[\{6\}, \frac{\pi}{3}\right] + \frac{35}{36}\pi^3\,\text{Zeta}[3] + \frac{15}{2}\pi\,\text{Zeta}[5], -\frac{74369\pi^7}{326592} + 135\,\text{Gl}\left[\{6, 1\}, \frac{\pi}{3}\right] - \frac{15}{2}\pi\,\text{Zeta}[3]^2\right\}$

Check numerically:

In[27]:=
```
Table[Ls[n, 0, Pi / 3] - LsToLi[Ls[n, 0, Pi / 3]], {n, 2, 7}] // N
```

Out[27]= $\{-3.33067\times 10^{-15}, 7.99361\times 10^{-15}, 1.36779\times 10^{-13}, -3.55271\times 10^{-14}, 1.49214\times 10^{-13}, -1.02318\times 10^{-12}\}$

■ A result of Zucker

In[28]:=
```
LsToLi[Ls[6, 3, Pi / 3] - 2 Ls[6, 1, Pi / 3]]
```

Out[28]= $\frac{313\pi^6}{204120}$

Reducing Nielsen polylogs at 1

In[29]:=
```
Li[{9, 1, 1, 1, 1, 1}, 1] // LiReduce
```

Out[29]=

$$\frac{5660117\,\pi^{14}}{980755776000} + \frac{49\,\pi^8\,\text{Zeta}[3]^2}{103680} + \frac{1}{36}\pi^2\,\text{Zeta}[3]^4 + \frac{59\,\pi^6\,\text{Zeta}[3]\,\text{Zeta}[5]}{5040} - \frac{11}{6}\text{Zeta}[3]^3\,\text{Zeta}[5] + \frac{4}{45}\pi^4\,\text{Zeta}[5]^2 + \frac{8}{45}\pi^4\,\text{Zeta}[3]\,\text{Zeta}[7] + \frac{11}{3}\pi^2\,\text{Zeta}[5]\,\text{Zeta}[7] - \frac{61\,\text{Zeta}[7]^2}{2} + \frac{35}{9}\pi^2\,\text{Zeta}[3]\,\text{Zeta}[9] - \frac{194}{3}\text{Zeta}[5]\,\text{Zeta}[9] - 72\,\text{Zeta}[3]\,\text{Zeta}[11]$$

This is based on K. S. Koelbig's expression (1982) for Nielsen polylogarithms at 1.

Numerical usage

In[30]:=
```
Ls[5, 2, 2 Pi / 3] // LsToLi
```

Out[30]=

$$-\frac{8\,\pi^5}{1215} - \frac{8}{9}\pi^2\,\text{Gl}\left[\{2, 1\}, \frac{2\,\pi}{3}\right] - \frac{8}{3}\pi\,\text{Gl}\left[\{3, 1\}, \frac{2\,\pi}{3}\right] + 4\,\text{Gl}\left[\{4, 1\}, \frac{2\,\pi}{3}\right]$$

In[31]:=
```
Ls[5, 2, 2 Pi / 3] // N
```

Out[31]= −0.518109

In[32]:=
```
N[Ls[5, 2, 2 Pi / 3] // LsToLi, 200]
```

Out[32]=
```
-0.51810878682968011734726563873169675502187966824315321406738947248246493059206791506817 5 \
917962342634092283168874070625727137897015228328288301238053344434601555482416349687142 6 \
4260545695615234087680879
```

For much higher precision, specialized code should be used for the evaluation of the multiple polylogarithms.

A random higher weight example

In[33]:=
```
N[Ls[15, 11, 2 Pi / 3] // LsToLi, 50]
```

Out[33]= −74.366995162588297902846888380312862221176197535466

Examples from "Log-sine evaluations of Mahler measures"

In[34]:=
```
Table[LsToLi[Ls[n, 0, Pi / 3] - Ls[n, 0, Pi]] / Pi, {n, 2, 7}] // Expand // Column
```

Out[34]=

$$\frac{\mathrm{Cl}\left[\{2\},\frac{\pi}{3}\right]}{\pi}$$
$$\frac{\pi^2}{54}$$
$$\frac{9\,\mathrm{Cl}\left[\{4\},\frac{\pi}{3}\right]}{2\pi}-\mathrm{Zeta}[3]$$
$$-\frac{\pi^4}{4860}+\frac{6\,\mathrm{Gl}\left[\{4,1\},\frac{\pi}{3}\right]}{\pi}$$
$$\frac{135\,\mathrm{Cl}\left[\{6\},\frac{\pi}{3}\right]}{2\pi}-\frac{5}{18}\pi^2\,\mathrm{Zeta}[3]-15\,\mathrm{Zeta}[5]$$
$$-\frac{943\,\pi^6}{40\,824}+\frac{135\,\mathrm{Gl}\left[\{6,1\},\frac{\pi}{3}\right]}{\pi}+15\,\mathrm{Zeta}[3]^2$$

Evaluation of μ (1+x)

In[35]:=
```
μk[k_] := -1 / Pi Ls[k + 1, 0, Pi] // LsToLi
```

In[36]:=
```
-μk[5]
μk[6]
```

Out[36]=
$$\frac{5}{4}\pi^2\,\mathrm{Zeta}[3]+\frac{45\,\mathrm{Zeta}[5]}{2}$$

Out[37]=
$$\frac{275\,\pi^6}{1344}+\frac{45\,\mathrm{Zeta}[3]^2}{2}$$

Reducibility

In[38]:=
```
$Assumptions = 0 < τ < Pi;
Ls[3, 0, τ] // LsToLi
Ls[3, 1, τ] // LsToLi
Ls[4, 0, τ] // LsToLi
Ls[4, 1, τ] // LsToLi
Ls[4, 2, τ] // LsToLi
```

Out[39]=
$$-\frac{\pi^2\,\tau}{4}+\frac{\pi\,\tau^2}{4}-\frac{\tau^3}{12}-2\,\mathrm{Gl}[\{2,1\},\tau]$$

Out[40]=
$$\tau\,\mathrm{Cl}[\{2\},\tau]+\mathrm{Cl}[\{3\},\tau]-\mathrm{Zeta}[3]$$

Out[41]=
$$\frac{3}{4}\pi^2\,\mathrm{Cl}[\{2\},\tau]-\frac{3}{2}\pi\,\tau\,\mathrm{Cl}[\{2\},\tau]+\frac{3}{4}\tau^2\,\mathrm{Cl}[\{2\},\tau]-\frac{3}{2}\pi\,\mathrm{Cl}[\{3\},\tau]+\frac{3}{2}\tau\,\mathrm{Cl}[\{3\},\tau]-\frac{3}{2}\mathrm{Cl}[\{4\},\tau]+6\,\mathrm{Cl}[\{2,1,1\},\tau]+\frac{3}{2}\pi\,\mathrm{Zeta}[3]$$

Out[42]=
$$\frac{\pi^4}{180}-\frac{\pi^2\,\tau^2}{8}+\frac{\pi\,\tau^3}{6}-\frac{\tau^4}{16}-2\,\tau\,\mathrm{Gl}[\{2,1\},\tau]-2\,\mathrm{Gl}[\{3,1\},\tau]$$

Out[43]=
$$\tau^2\,\mathrm{Cl}[\{2\},\tau]+2\,\tau\,\mathrm{Cl}[\{3\},\tau]-2\,\mathrm{Cl}[\{4\},\tau]$$

Appendix A from "New results for the ϵ -expansionof certain one-, two- and three-loop Feynman diagrams" by A. Davydychev and M. Kalmykov

In[44]:=
```
$Assumptions = 0 < τ < Pi;
Ls[3, 1, τ] // LsToLi
Ls[4, 2, τ] // LsToLi
Ls[5, 3, τ] // LsToLi
Ls[6, 4, τ] // LsToLi
```

Out[45]= $\tau\,\mathrm{Cl}[\{2\},\tau] + \mathrm{Cl}[\{3\},\tau] - \mathrm{Zeta}[3]$

Out[46]= $\tau^2\,\mathrm{Cl}[\{2\},\tau] + 2\,\tau\,\mathrm{Cl}[\{3\},\tau] - 2\,\mathrm{Cl}[\{4\},\tau]$

Out[47]= $\tau^3\,\mathrm{Cl}[\{2\},\tau] + 3\,\tau^2\,\mathrm{Cl}[\{3\},\tau] - 6\,\tau\,\mathrm{Cl}[\{4\},\tau] - 6\,\mathrm{Cl}[\{5\},\tau] + 6\,\mathrm{Zeta}[5]$

Out[48]= $\tau^4\,\mathrm{Cl}[\{2\},\tau] + 4\,\tau^3\,\mathrm{Cl}[\{3\},\tau] - 12\,\tau^2\,\mathrm{Cl}[\{4\},\tau] - 24\,\tau\,\mathrm{Cl}[\{5\},\tau] + 24\,\mathrm{Cl}[\{6\},\tau]$

In[49]:=
```
Ls[2, 0, Pi] // LsToLi
Ls[3, 0, Pi] // LsToLi
Ls[4, 0, Pi] // LsToLi
Ls[5, 0, Pi] // LsToLi
Ls[6, 0, Pi] // LsToLi
```

Out[49]= 0

Out[50]= $-\frac{\pi^3}{12}$

Out[51]= $\frac{3}{2}\,\pi\,\mathrm{Zeta}[3]$

Out[52]= $-\frac{19\,\pi^5}{240}$

Out[53]= $\frac{5}{4}\,\pi^3\,\mathrm{Zeta}[3] + \frac{45}{2}\,\pi\,\mathrm{Zeta}[5]$

In[54]:=
```
Ls[3, 0, Pi / 3] // LsToLi
Ls[4, 1, Pi / 3] // LsToLi
```

Out[54]= $-\frac{7\,\pi^3}{108}$

Out[55]= $-\frac{17\,\pi^4}{6480}$

- **(A.9)**

In[56]:=
```
Ls[4, 0, Pi / 3] // LsToLi
Ls[6, 0, Pi / 3] // LsToLi
```

Out[56]= $\frac{9}{2}\,\mathrm{Cl}\left[\{4\},\frac{\pi}{3}\right] + \frac{1}{2}\,\pi\,\mathrm{Zeta}[3]$

Out[57]= $\frac{135}{2}\,\mathrm{Cl}\left[\{6\},\frac{\pi}{3}\right] + \frac{35}{36}\,\pi^3\,\mathrm{Zeta}[3] + \frac{15}{2}\,\pi\,\mathrm{Zeta}[5]$

- **(A.10)**

In[58]:=
```
Ls[5, 1, Pi / 3] - Pi / 3 Ls[4, 0, Pi / 3] // LsToLi
Ls[6, 1, Pi / 3] - Pi / 3 Ls[5, 0, Pi / 3] // LsToLi
Ls[7, 1, Pi / 3] - Pi / 3 Ls[6, 0, Pi / 3] // LsToLi
Ls[8, 1, Pi / 3] - Pi / 3 Ls[7, 0, Pi / 3] // LsToLi
```

Out[58]= $-\frac{1}{12}\pi^2\,\mathrm{Zeta}[3]-\frac{19\,\mathrm{Zeta}[5]}{4}$

Out[59]= $\frac{2029\,\pi^6}{90\,720}+2\,\mathrm{Zeta}[3]^2$

Out[60]= $-\frac{41}{144}\pi^4\,\mathrm{Zeta}[3]-\frac{5}{4}\pi^2\,\mathrm{Zeta}[5]-\frac{2465\,\mathrm{Zeta}[7]}{32}$

Out[61]= $\frac{1\,080\,479\,\pi^8}{13\,063\,680}-405\,\mathrm{Gl}\left[\{7,1\},\frac{\pi}{3}\right]+\frac{5}{4}\pi^2\,\mathrm{Zeta}[3]^2-45\,\mathrm{Zeta}[3]\,\mathrm{Zeta}[5]$

Note that the built-in reductions only reduce among polylogarithms of Nielsen type. That's why Gl[{7,1},pi/3] is not reduced to zeta values including zeta(5,3).

In[62]:=
```
Ls[9, 1, Pi / 3] - Pi / 3 Ls[8, 0, Pi / 3] // LsToLi
```

Out[62]= $-\frac{2029\,\pi^6\,\mathrm{Zeta}[3]}{1728}-35\,\mathrm{Zeta}[3]^3-\frac{287}{32}\pi^4\,\mathrm{Zeta}[5]-\frac{315}{8}\pi^2\,\mathrm{Zeta}[7]-\frac{487\,235\,\mathrm{Zeta}[9]}{192}$

- **(A.11)**

In[63]:=
```
Ls[5, 2, Pi / 3] - 2 / 3 Ls[5, 0, Pi / 3] // LsToLi
Ls[6, 2, Pi / 3] + 4 / 15 Ls[6, 0, Pi / 3] - 2 / 3 Zeta[2] Ls[4, 0, Pi / 3] // LsToLi
Ls[6, 3, Pi / 3] - 2 / 3 Pi Ls[5, 0, Pi / 3] // LsToLi
```

Out[63]= $\frac{253\,\pi^5}{4860}$

Out[64]= $\frac{11}{18}\pi^3\,\mathrm{Zeta}[3]+\frac{5}{6}\pi\,\mathrm{Zeta}[5]$

Out[65]= $\frac{18\,887\,\pi^6}{408\,240}+4\,\mathrm{Zeta}[3]^2$

- **(A.14)**

Our package will provide evaluations in terms of Clausen/Glaisher values at pi/2. These can be rewritten as polylogarithms at 1/2.

In[66]:=
```
Ls[4, 1, Pi / 2] // LsToLi
Ls[3, 0, Pi / 2] // LsToLi
```

Out[66]= $-\frac{101\,\pi^4}{11\,520}-\pi\,\mathrm{Gl}\left[\{2,1\},\frac{\pi}{2}\right]-2\,\mathrm{Gl}\left[\{3,1\},\frac{\pi}{2}\right]$

Out[67]= $-\frac{7\,\pi^3}{96}-2\,\mathrm{Gl}\left[\{2,1\},\frac{\pi}{2}\right]$

In[68]:= `Ls[5, 1, Pi / 2] // LsToLi`

Out[68]= $\frac{3}{32}\pi^3\,\mathrm{Cl}\left[\{2\}, \frac{\pi}{2}\right] + \frac{3}{4}\pi\,\mathrm{Cl}\left[\{4\}, \frac{\pi}{2}\right] + 3\pi\,\mathrm{Cl}\left[\{2, 1, 1\}, \frac{\pi}{2}\right] + 6\,\mathrm{Cl}\left[\{3, 1, 1\}, \frac{\pi}{2}\right] + \frac{137}{512}\pi^2\,\mathrm{Zeta}[3] - \frac{7545\,\mathrm{Zeta}[5]}{1024}$

Again, the alternating zeta values can be rewritten as polylogarithms at 1/2.

In[69]:=
```
Ls[4, 1, Pi] // LsToLi
Ls[5, 1, Pi] // LsToLi
Ls[5, 2, Pi] // LsToLi
```

Out[69]= $-\frac{11\pi^4}{720} - 2\,\mathrm{Li}[\{3, 1\}, -1]$

Out[70]= $6\,\mathrm{Li}[\{3, 1, 1\}, -1] + \frac{1}{4}\pi^2\,\mathrm{Zeta}[3] - \frac{105\,\mathrm{Zeta}[5]}{32}$

Out[71]= $-\frac{\pi^5}{120} - 4\pi\,\mathrm{Li}[\{3, 1\}, -1]$

- **Numerical values**

In[72]:= `N[Ls[3, 0, Pi / 2] // LsToLi, 100]`

Out[72]= −2.033576506072054600912068969700518249992376075613046185506487429858439689686915123555411633065963200

In[73]:= `N[Ls[4, 0, Pi / 2] // LsToLi, 100]`

Out[73]= 6.00310955652900656730930561403268855953045335060113598674445599216404336646499663878885196889407506 7

In[74]:= `N[Cl[{4}, Pi / 2], 100]`

Out[74]= 0.988944551741105336108422633228377821315860887062733910781992401639015194698018196411910468999 7999338

In[75]:= `N[Ls[5, 0, Pi / 2] // LsToLi, 100]`

Out[75]= −24.01433772015983592359467991814460623522190819473543320027555234179026210338733065095551060837656047

In[76]:= `N[Ls[5, 2, Pi / 2] // LsToLi, 100]`

Out[76]= −0.12681324283558869710023229966108992518388946762736056047073509227972281612763856489996414941 75549643

Examples from " Explicit evaluations of some families of log-sine and log-cosine integrals"

In[77]:=
```
Table[LsToLi[-Ls[n, 1, 2 Pi]], {n, 4, 12}] // Column
```

Out[77]=

$\frac{\pi^4}{6}$

$-3\pi^2\,\text{Zeta}[3]$

$\frac{19\pi^6}{120}$

$-\frac{5}{2}\pi^4\,\text{Zeta}[3]-45\pi^2\,\text{Zeta}[5]$

$\frac{275\pi^8}{672}+45\pi^2\,\text{Zeta}[3]^2$

$-\frac{133}{16}\pi^6\,\text{Zeta}[3]-\frac{315}{4}\pi^4\,\text{Zeta}[5]-\frac{2835}{2}\pi^2\,\text{Zeta}[7]$

$\frac{11813\pi^{10}}{5760}+105\pi^4\,\text{Zeta}[3]^2+3780\pi^2\,\text{Zeta}[3]\,\text{Zeta}[5]$

$-\frac{825}{16}\pi^8\,\text{Zeta}[3]-1890\pi^2\,\text{Zeta}[3]^3-\frac{3591}{8}\pi^6\,\text{Zeta}[5]-\frac{8505}{2}\pi^4\,\text{Zeta}[7]-80325\pi^2\,\text{Zeta}[9]$

$\frac{95265\pi^{12}}{5632}+\frac{5985}{8}\pi^6\,\text{Zeta}[3]^2+14175\pi^4\,\text{Zeta}[3]\,\text{Zeta}[5]+127575\pi^2\,\text{Zeta}[5]^2+255150\pi^2\,\text{Zeta}[3]\,\text{Zeta}[7]$

Last three are given incorrectly in the reference.

In[78]:=
```
LsToLi[-Ls[10, 1, 2 Pi]]
N[%, 100]
N[82 691 / 40 320 Pi^10 + 3780 Pi^2 Zeta[3] Zeta[5] + 315 / 4 Pi^4 Zeta[3]^2, 100]
NIntegrate[t (Log[2 Sin[t / 2]])^8, {t, 0, 2 Pi}]
```

Out[78]= $\frac{11813\pi^{10}}{5760}+105\pi^4\,\text{Zeta}[3]^2+3780\pi^2\,\text{Zeta}[3]\,\text{Zeta}[5]$

Out[79]= 253 339.875403818574479263306500484081690169969375098863175713426527868700507180787704808866 96624831376

Out[80]= 249 645.178197267715841033773523137558849639655041722877072356018542142129884004579584897115 9386618080

Out[81]= 253 340.

In[82]:=
```
Table[LsToLi[-Ls[n, 2, 2 Pi]], {n, 4, 13}] // Column
```

Out[82]=

$-4\pi\,\text{Zeta}[3]$

$\frac{13\pi^5}{45}$

$-5\pi^3\,\text{Zeta}[3]-12\pi\,\text{Zeta}[5]$

$\frac{29\pi^7}{105}+24\pi\,\text{Zeta}[3]^2$

$-\frac{71}{12}\pi^5\,\text{Zeta}[3]-70\pi^3\,\text{Zeta}[5]-90\pi\,\text{Zeta}[7]$

$\frac{3517\pi^9}{5040}+90\pi^3\,\text{Zeta}[3]^2+900\pi\,\text{Zeta}[3]\,\text{Zeta}[5]$

$-\frac{971}{48}\pi^7\,\text{Zeta}[3]-630\pi\,\text{Zeta}[3]^3-\frac{679}{4}\pi^5\,\text{Zeta}[5]-\frac{4095}{2}\pi^3\,\text{Zeta}[7]-1260\pi\,\text{Zeta}[9]$

$\frac{54343\pi^{11}}{15840}+315\pi^5\,\text{Zeta}[3]^2+7140\pi^3\,\text{Zeta}[3]\,\text{Zeta}[5]+15120\pi\,\text{Zeta}[5]^2+30240\pi\,\text{Zeta}[3]\,\text{Zeta}[7]$

$-\frac{39949}{320}\pi^9\,\text{Zeta}[3]-4410\pi^3\,\text{Zeta}[3]^3-\frac{3957}{4}\pi^7\,\text{Zeta}[5]-$
$90720\pi\,\text{Zeta}[3]^2\,\text{Zeta}[5]-\frac{33075}{4}\pi^5\,\text{Zeta}[7]-110880\pi^3\,\text{Zeta}[9]-28350\pi\,\text{Zeta}[11]$

$\frac{35803881\pi^{13}}{1281280}+\frac{9345}{4}\pi^7\,\text{Zeta}[3]^2+37800\pi\,\text{Zeta}[3]^4+\frac{79065}{2}\pi^5\,\text{Zeta}[3]\,\text{Zeta}[5]+226800\pi^3\,\text{Zeta}[5]^2+$
$453600\pi^3\,\text{Zeta}[3]\,\text{Zeta}[7]+1530900\pi\,\text{Zeta}[5]\,\text{Zeta}[7]+1833300\pi\,\text{Zeta}[3]\,\text{Zeta}[9]$

In[83]:=
```
Table[LsToLi[-Ls[n, 3, 2 Pi]], {n, 5, 14}] // Column
```

Out[83]=

$$-12\,\pi^2\,\text{Zeta}[3]$$

$$\frac{8\,\pi^6}{15}$$

$$-9\,\pi^4\,\text{Zeta}[3]-36\,\pi^2\,\text{Zeta}[5]$$

$$\frac{43\,\pi^8}{84}+72\,\pi^2\,\text{Zeta}[3]^2$$

$$-\frac{51}{4}\,\pi^6\,\text{Zeta}[3]-120\,\pi^4\,\text{Zeta}[5]-270\,\pi^2\,\text{Zeta}[7]$$

$$\frac{51\,\pi^{10}}{40}+180\,\pi^4\,\text{Zeta}[3]^2+2700\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[5]$$

$$-\frac{705}{16}\,\pi^8\,\text{Zeta}[3]-1890\,\pi^2\,\text{Zeta}[3]^3-\frac{1407}{4}\,\pi^6\,\text{Zeta}[5]-\frac{6615}{2}\,\pi^4\,\text{Zeta}[7]-3780\,\pi^2\,\text{Zeta}[9]$$

$$\frac{39\,223\,\pi^{12}}{6336}+735\,\pi^6\,\text{Zeta}[3]^2+13\,860\,\pi^4\,\text{Zeta}[3]\,\text{Zeta}[5]+45\,360\,\pi^2\,\text{Zeta}[5]^2+90\,720\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[7]$$

$$-\frac{86\,847}{320}\,\pi^{10}\,\text{Zeta}[3]-9450\,\pi^4\,\text{Zeta}[3]^3-2070\,\pi^8\,\text{Zeta}[5]-272\,160\,\pi^2\,\text{Zeta}[3]^2\,\text{Zeta}[5]-\frac{65\,205}{4}\,\pi^6\,\text{Zeta}[7]-171\,990\,\pi^4\,\text{Zeta}[9]-85\,050\,\pi^2\,\text{Zeta}[11]$$

$$\frac{1\,456\,047\,\pi^{14}}{29\,120}+\frac{11\,025}{2}\,\pi^8\,\text{Zeta}[3]^2+113\,400\,\pi^2\,\text{Zeta}[3]^4+\frac{180\,495}{2}\,\pi^6\,\text{Zeta}[3]\,\text{Zeta}[5]+425\,250\,\pi^4\,\text{Zeta}[5]^2+850\,500\,\pi^4\,\text{Zeta}[3]\,\text{Zeta}[7]+4\,592\,700\,\pi^2\,\text{Zeta}[5]\,\text{Zeta}[7]+5\,499\,900\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[9]$$

In[84]:=
```
Table[LsToLi[-Ls[n, 4, 2 Pi]], {n, 5, 15}] // Column
```

Out[84]=

$$\frac{32\,\pi^5}{5}$$

$$-32\,\pi^3\,\text{Zeta}[3]+48\,\pi\,\text{Zeta}[5]$$

$$\frac{296\,\pi^7}{315}+48\,\pi\,\text{Zeta}[3]^2$$

$$-20\,\pi^5\,\text{Zeta}[3]-84\,\pi^3\,\text{Zeta}[5]+360\,\pi\,\text{Zeta}[7]$$

$$\frac{457\,\pi^9}{525}+216\,\pi^3\,\text{Zeta}[3]^2+288\,\pi\,\text{Zeta}[3]\,\text{Zeta}[5]$$

$$-\frac{92}{3}\,\pi^7\,\text{Zeta}[3]-720\,\pi\,\text{Zeta}[3]^3-229\,\pi^5\,\text{Zeta}[5]-420\,\pi^3\,\text{Zeta}[7]+5040\,\pi\,\text{Zeta}[9]$$

$$\frac{3253\,\pi^{11}}{1540}+513\,\pi^5\,\text{Zeta}[3]^2+7560\,\pi^3\,\text{Zeta}[3]\,\text{Zeta}[5]-2160\,\pi\,\text{Zeta}[5]^2-4320\,\pi\,\text{Zeta}[3]\,\text{Zeta}[7]$$

$$-\frac{6259}{60}\,\pi^9\,\text{Zeta}[3]-6300\,\pi^3\,\text{Zeta}[3]^3-\frac{2893}{4}\,\pi^7\,\text{Zeta}[5]-45\,360\,\pi\,\text{Zeta}[3]^2\,\text{Zeta}[5]-\frac{10\,353}{2}\,\pi^5\,\text{Zeta}[7]-1260\,\pi^3\,\text{Zeta}[9]+113\,400\,\pi\,\text{Zeta}[11]$$

$$\frac{9\,155\,089\,\pi^{13}}{900\,900}+2259\,\pi^7\,\text{Zeta}[3]^2+30\,240\,\pi\,\text{Zeta}[3]^4+35\,532\,\pi^5\,\text{Zeta}[3]\,\text{Zeta}[5]+115\,920\,\pi^3\,\text{Zeta}[5]^2+231\,840\,\pi^3\,\text{Zeta}[3]\,\text{Zeta}[7]-544\,320\,\pi\,\text{Zeta}[5]\,\text{Zeta}[7]-302\,400\,\pi\,\text{Zeta}[3]\,\text{Zeta}[9]$$

$$-\frac{25\,467}{40}\,\pi^{11}\,\text{Zeta}[3]-32\,886\,\pi^5\,\text{Zeta}[3]^3-\frac{330\,801}{80}\,\pi^9\,\text{Zeta}[5]-861\,840\,\pi^3\,\text{Zeta}[3]^2\,\text{Zeta}[5]-1\,360\,800\,\pi\,\text{Zeta}[3]\,\text{Zeta}[5]^2-\frac{54\,369}{2}\,\pi^7\,\text{Zeta}[7]-1\,360\,800\,\pi\,\text{Zeta}[3]^2\,\text{Zeta}[7]-246\,078\,\pi^5\,\text{Zeta}[9]+113\,400\,\pi^3\,\text{Zeta}[11]+3\,742\,200\,\pi\,\text{Zeta}[13]$$

$$\frac{9\,855\,431\,\pi^{15}}{120\,120}+\frac{270\,063}{16}\,\pi^9\,\text{Zeta}[3]^2+415\,800\,\pi^3\,\text{Zeta}[3]^4+248\,895\,\pi^7\,\text{Zeta}[3]\,\text{Zeta}[5]+6\,350\,400\,\pi\,\text{Zeta}[3]^3\,\text{Zeta}[5]+962\,010\,\pi^5\,\text{Zeta}[5]^2+1\,924\,020\,\pi^5\,\text{Zeta}[3]\,\text{Zeta}[7]+10\,206\,000\,\pi^3\,\text{Zeta}[5]\,\text{Zeta}[7]-27\,556\,200\,\pi\,\text{Zeta}[7]^2+13\,532\,400\,\pi^3\,\text{Zeta}[3]\,\text{Zeta}[9]-42\,411\,600\,\pi\,\text{Zeta}[5]\,\text{Zeta}[9]-17\,010\,000\,\pi\,\text{Zeta}[3]\,\text{Zeta}[11]$$

In[85]:=
```
Table[LsToLi[-Ls[n, 5, 2 Pi]], {n, 7, 16}] // Column
```

Out[85]=

$$-80\,\pi^4\,\text{Zeta}[3]+240\,\pi^2\,\text{Zeta}[5]$$

$$\frac{100\,\pi^8}{63}+240\,\pi^2\,\text{Zeta}[3]^2$$

$$-48\,\pi^6\,\text{Zeta}[3]-180\,\pi^4\,\text{Zeta}[5]+1800\,\pi^2\,\text{Zeta}[7]$$

$$\frac{143\,\pi^{10}}{105}+600\,\pi^4\,\text{Zeta}[3]^2+1440\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[5]$$

$$-75\,\pi^8\,\text{Zeta}[3]-3600\,\pi^2\,\text{Zeta}[3]^3-465\,\pi^6\,\text{Zeta}[5]-300\,\pi^4\,\text{Zeta}[7]+25\,200\,\pi^2\,\text{Zeta}[9]$$

$$\frac{2185\,\pi^{12}}{693}+1485\,\pi^6\,\text{Zeta}[3]^2+19\,800\,\pi^4\,\text{Zeta}[3]\,\text{Zeta}[5]-10\,800\,\pi^2\,\text{Zeta}[5]^2-21\,600\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[7]$$

$$-250\,\pi^{10}\,\text{Zeta}[3]-18\,900\,\pi^4\,\text{Zeta}[3]^3-\frac{5925}{4}\,\pi^8\,\text{Zeta}[5]-226\,800\,\pi^2\,\text{Zeta}[3]^2\,\text{Zeta}[5]-\frac{15\,225}{2}\,\pi^6\,\text{Zeta}[7]+18\,900\,\pi^4\,\text{Zeta}[9]+567\,000\,\pi^2\,\text{Zeta}[11]$$

$$\frac{2\,704\,463\,\pi^{14}}{180\,180}+6675\,\pi^8\,\text{Zeta}[3]^2+151\,200\,\pi^2\,\text{Zeta}[3]^4+95\,340\,\pi^6\,\text{Zeta}[3]\,\text{Zeta}[5]+277\,200\,\pi^4\,\text{Zeta}[5]^2+554\,400\,\pi^4\,\text{Zeta}[3]\,\text{Zeta}[7]-2\,721\,600\,\pi^2\,\text{Zeta}[5]\,\text{Zeta}[7]-1\,512\,000\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[9]$$

$$-\frac{24\,185}{16}\,\pi^{12}\,\text{Zeta}[3]-106\,470\,\pi^6\,\text{Zeta}[3]^3-\frac{129\,153}{16}\,\pi^{10}\,\text{Zeta}[5]-2\,494\,800\,\pi^4\,\text{Zeta}[3]^2\,\text{Zeta}[5]-6\,804\,000\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[5]^2-\frac{77\,175}{2}\,\pi^8\,\text{Zeta}[7]-6\,804\,000\,\pi^2\,\text{Zeta}[3]^2\,\text{Zeta}[7]-297\,990\,\pi^6\,\text{Zeta}[9]+1\,134\,000\,\pi^4\,\text{Zeta}[11]+18\,711\,000\,\pi^2\,\text{Zeta}[13]$$

$$\frac{23\,446\,495\,\pi^{16}}{192\,192}+\frac{794\,235}{16}\,\pi^{10}\,\text{Zeta}[3]^2+1\,323\,000\,\pi^4\,\text{Zeta}[3]^4+680\,625\,\pi^8\,\text{Zeta}[3]\,\text{Zeta}[5]+31\,752\,000\,\pi^2\,\text{Zeta}[3]^3\,\text{Zeta}[5]+2\,315\,250\,\pi^6\,\text{Zeta}[5]^2+4\,630\,500\,\pi^6\,\text{Zeta}[3]\,\text{Zeta}[7]+20\,412\,000\,\pi^4\,\text{Zeta}[5]\,\text{Zeta}[7]-137\,781\,000\,\pi^2\,\text{Zeta}[7]^2+30\,996\,000\,\pi^4\,\text{Zeta}[3]\,\text{Zeta}[9]-212\,058\,000\,\pi^2\,\text{Zeta}[5]\,\text{Zeta}[9]-85\,050\,000\,\pi^2\,\text{Zeta}[3]\,\text{Zeta}[11]$$

In[86]:=
```
Table[LsToLi[-Ls[n, 6, 2 Pi]], {n, 8, 12}] // Column
```

Out[86]=

$$-192\,\pi^5\,\text{Zeta}[3]+960\,\pi^3\,\text{Zeta}[5]-1440\,\pi\,\text{Zeta}[7]$$

$$\frac{856\,\pi^9}{315}+960\,\pi^3\,\text{Zeta}[3]^2-2880\,\pi\,\text{Zeta}[3]\,\text{Zeta}[5]$$

$$-112\,\pi^7\,\text{Zeta}[3]-1440\,\pi\,\text{Zeta}[3]^3-264\,\pi^5\,\text{Zeta}[5]+6840\,\pi^3\,\text{Zeta}[7]-20\,160\,\pi\,\text{Zeta}[9]$$

$$\frac{7936\,\pi^{11}}{3465}+1776\,\pi^5\,\text{Zeta}[3]^2+4320\,\pi^3\,\text{Zeta}[3]\,\text{Zeta}[5]-17\,280\,\pi\,\text{Zeta}[5]^2-34\,560\,\pi\,\text{Zeta}[3]\,\text{Zeta}[7]$$

$$-\frac{514}{3}\,\pi^9\,\text{Zeta}[3]-15\,600\,\pi^3\,\text{Zeta}[3]^3-760\,\pi^7\,\text{Zeta}[5]+2910\,\pi^5\,\text{Zeta}[7]+84\,000\,\pi^3\,\text{Zeta}[9]-453\,600\,\pi\,\text{Zeta}[11]$$